The Annals of Probability
2015, Vol. 43, No. 2, 875–933
DOI: 10.1214/14-AOP880

A COPOLYMER NEAR A SELECTIVE INTERFACE: VARIATIONAL CHARACTERIZATION OF THE FREE ENERGY[1]

BY ERWIN BOLTHAUSEN[2], FRANK DEN HOLLANDER[3] AND ALEX A. OPOKU[4]

Universität Zürich, Leiden University and Delft University of Technology

In this paper, we consider a random copolymer near a selective interface separating two solvents. The configurations of the copolymer are directed paths that can make i.i.d. excursions of finite length above and below the interface. The excursion length distribution is assumed to have a tail that is logarithmically equivalent to a power law with exponent $\alpha \geq 1$. The monomers carry i.i.d. real-valued types whose distribution is assumed to have zero mean, unit variance, and a finite moment generating function. The interaction Hamiltonian rewards matches and penalizes mismatches of the monomer types and the solvents, and depends on two parameters: the interaction strength $\beta \geq 0$ and the interaction bias $h \geq 0$. We are interested in the behavior of the copolymer in the limit as its length tends to infinity.

The quenched free energy per monomer $(\beta, h) \mapsto g^{\text{que}}(\beta, h)$ has a phase transition along a quenched critical curve $\beta \mapsto h_c^{\text{que}}(\beta)$ separating a localized phase, where the copolymer stays close to the interface, from a delocalized phase, where the copolymer wanders away from the interface. We derive *variational formulas* for both these quantities. We compare these variational formulas with their analogues for the annealed free energy per monomer $(\beta, h) \mapsto g^{\text{ann}}(\beta, h)$ and the annealed critical curve $\beta \mapsto h_c^{\text{ann}}(\beta)$, both of which are explicitly computable. This comparison leads to:

(1) A proof that $g^{\text{que}}(\beta, h) < g^{\text{ann}}(\beta, h)$ for all $\alpha \geq 1$ and (β, h) in the annealed localized phase.

(2) A proof that $h_c^{\text{ann}}(\beta/\alpha) < h_c^{\text{que}}(\beta) < h_c^{\text{ann}}(\beta)$ for all $\alpha > 1$ and $\beta > 0$.

(3) A proof that $\liminf_{\beta \downarrow 0} h_c^{\text{que}}(\beta)/\beta \geq (1 + \alpha)/2\alpha$ for all $\alpha \geq 2$.

(4) A proof that $\liminf_{\beta \downarrow 0} h_c^{\text{que}}(\beta)/\beta \geq K_c^*(\alpha)$ for all $1 < \alpha < 2$ with $K_c^*(\alpha)$ given by an explicit integral criterion.

(5) An upper bound on the total number of times the copolymer visits the interface in the interior of the quenched delocalized phase.

Received November 2012.

[1]This work was done while A. A. Opoku was a postdoc at the Mathematical Institute of Leiden University during the period January 2010 until December 2012.

[2]Supported by SNSF-Grant 20-100536/1.

[3]Supported by ERC Advanced Grant VARIS 267356.

[4]Supported by NWO-Grant 613.000.913.

MSC2010 subject classifications. Primary 60F10, 60K37; secondary 82B27, 82B44.

Key words and phrases. Copolymer, selective interface, free energy, critical curve, localization vs delocalization, large deviation principle, variational formula, specific relative entropy.

(6) An identification of the asymptotic frequency at which the copolymer visits the interface in the quenched localized phase.

The copolymer model has been studied extensively in the literature. The goal of the present paper is to open up a window with a variational view and to settle a number of open problems.

1. Introduction and main results.

In Section 1.1, we define the model. In Sections 1.2 and 1.3, we define the quenched and the annealed free energy and critical curve. In Section 1.4, we state our main results, while in Section 1.5 we place these results in the context of earlier work. In Section 1.6, we give an outline of the rest of the paper.

For more background and key results in the literature, we refer the reader to Giacomin [21], Chapters 6–8, and den Hollander [16], Chapter 9.

1.1. *A copolymer near a selective interface.*

Throughout the paper, $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\omega = (\omega_k)_{k\in\mathbb{N}}$ be i.i.d. random variables with a probability distribution ν on $\mathbb{R}$ having zero mean and unit variance:

$$\int_{\mathbb{R}} x\,\nu(dx) = 0, \qquad \int_{\mathbb{R}} x^2\nu(dx) = 1 \tag{1.1}$$

and a finite cumulant generating function:

$$M(\lambda) = \log \int_{\mathbb{R}} e^{-\lambda x}\nu(dx) < \infty \qquad \forall \lambda \in \mathbb{R}. \tag{1.2}$$

Write $\mathbb{P} = \nu^{\otimes\mathbb{N}}$ to denote the distribution of ω. Let

$$\begin{aligned}\Pi = \big\{\pi = (k, \pi_k)_{k\in\mathbb{N}_0} : \pi_0 = 0, \operatorname{sign}(\pi_{k-1}) + \operatorname{sign}(\pi_k) \neq 0,\\ \pi_k \in \mathbb{Z}\ \forall k \in \mathbb{N}\big\},\end{aligned} \tag{1.3}$$

where $\operatorname{sign}(\pi_k) \in \{-1, 0, +1\}$ depending on whether π_k is below, on or above the interface $\mathbb{N}_0 \times \{0\}$. In words, Π is the set of infinite directed paths on $\mathbb{N}_0 \times \mathbb{Z}$ that start at the origin and, when crossing over from the lower half-plane to the upper half-plane, or vice versa, hit the interface once. Fix $n \in \mathbb{N}_0$ and $\beta, h \geq 0$. For given ω, let

$$H_n^{\beta,h,\omega}(\pi) = -\beta \sum_{k=1}^{n} (\omega_k + h) \operatorname{sign}(\pi_{k-1}, \pi_k), \qquad \pi \in \Pi, \tag{1.4}$$

be the n-step Hamiltonian on Π, where $\Delta_k = \operatorname{sign}(\pi_{k-1}, \pi_k) \in \{-1, +1\}$ (the kth edge lies below or above the interface), and let

$$\frac{dP_n^{\beta,h,\omega}}{dP}(\pi) = \frac{1}{Z_n^{\beta,h,\omega}} e^{-H_n^{\beta,h,\omega}(\pi)}, \qquad \pi \in \Pi, \tag{1.5}$$

be the n-step path measure on Π, where P is any probability distribution on Π under which the excursions away from the interface are i.i.d., lie with probability $\frac{1}{2}$ below and above the interface, and have a length whose probability distribution ρ on $\mathbb{N}$ has a *polynomial tail*

$$\lim_{\substack{m\to\infty\\ \rho(m)>0}} \frac{\log \rho(m)}{\log m} = -\alpha \qquad \text{for some } \alpha \geq 1. \tag{1.6}$$

A mild regularity condition is needed to control the *sparsity* of the support of ρ, namely,

$$\lim_{n\to\infty} \frac{1}{n} \log\left[\sum_{m>n} \rho(m)\right] = 0. \tag{1.7}$$

Note that the Hamiltonian in (1.4) only depends on the signs of the excursions, not on their shape. In (1.3), excursions of length 1 are excluded (but could be easily included as well).

EXAMPLE. For the special case where ν is the binary distribution $\nu(-1) = \nu(+1) = \frac{1}{2}$ and P is simple random walk on $\mathbb{Z}$, the above definitions have the following interpretation (see Figure 1). Think of $\pi \in \Pi$ in (1.3) as the path of a directed copolymer on $\mathbb{N}_0 \times \mathbb{Z}$, consisting of monomers represented by the edges (π_{k-1}, π_k), $k \in \mathbb{N}$, pointing either north–east of south–east. Think of the lower half-plane as water and the upper half-plane as oil. The monomers are labeled by ω, with $\omega_k = -1$ indicating that monomer k is hydrophilic and $\omega_k = +1$ that it is hydrophobic. Both types occur with density $\frac{1}{2}$. The factor $\mathrm{sign}(\pi_{k-1}, \pi_k)$ in (1.4) equals -1 or $+1$ depending on whether monomer k lies in the water or in the oil. The interaction Hamiltonian in (1.4) therefore rewards matches and penalizes mismatches of the monomer types and the solvents. The parameter β is the *interaction strength* (or inverse temperature), the parameter h plays the role of the *interaction bias*: $h = 0$ corresponds to the hydrophobic and hydrophilic monomers interacting

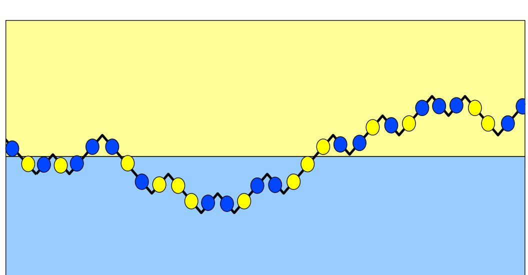

FIG. 1. *A directed copolymer near a linear interface. Oil in the upper half-plane and hydrophobic monomers in the polymer chain are shaded light, water in the lower half-plane and hydrophilic monomers in the polymer chain are shaded dark.* (*Courtesy of N. Pétrélis.*)

equally strongly, while $h = 1$ corresponds to the hydrophilic monomers not interacting at all. The probability distribution of the copolymer given ω is the quenched Gibbs distribution in (1.5). For simple random walk, the support of ρ is $2\mathbb{N}$ and the exponent is $\alpha = \frac{3}{2}: \rho(2m) \sim 1/2\pi^{1/2}m^{3/2}$ as $m \to \infty$ (Feller [19], Chapter III).

1.2. *Quenched free energy and critical curve.* The model in Section 1.1 was introduced in Garel, Huse, Leibler and Orland [20]. It was shown in Bolthausen and den Hollander [9] that for every $\beta, h \geq 0$ the *quenched free energy* per monomer

$$f^{\text{que}}(\beta, h) = \lim_{n\to\infty} \frac{1}{n} \log Z_n^{\beta,h,\omega} \tag{1.8}$$

exists ω-a.s. and in $L^1(\mathbb{P})$, and is ω-a.s. constant.

It was further noted that

$$f^{\text{que}}(\beta, h) \geq \beta h. \tag{1.9}$$

This lower bound comes from the strategy where the path spends all of its time above the interface, that is, $\pi_k > 0$ for $1 \leq k \leq n$. Indeed, in that case $\text{sign}(\pi_{k-1}, \pi_k) = +1$ for $1 \leq k \leq n$, resulting in $H_n^{\beta,h,\omega}(\pi) = -\beta h n[1 + o(1)]$ ω-a.s. as $n \to \infty$ by the strong law of large numbers for ω [recall (1.1)]. Since $P(\{\pi \in \Pi : \pi_k > 0 \text{ for } 1 \leq k \leq n\}) = \sum_{k>n} \rho(k)$, the cost of this strategy under P is negligible on an exponential scale by (1.7).

In view of (1.9), it is natural to introduce the *quenched excess free energy*

$$g^{\text{que}}(\beta, h) = f^{\text{que}}(\beta, h) - \beta h, \tag{1.10}$$

to define the two phases

$$\begin{aligned} \mathcal{D}^{\text{que}} &= \{(\beta, h) : g^{\text{que}}(\beta, h) = 0\}, \\ \mathcal{L}^{\text{que}} &= \{(\beta, h) : g^{\text{que}}(\beta, h) > 0\} \end{aligned} \tag{1.11}$$

and to refer to $\mathcal{D}^{\text{que}}$ as the *quenched delocalized phase*, where the strategy of staying above the interface is optimal (at the level of free energy), and to $\mathcal{L}^{\text{que}}$ as the *quenched localized phase*, where this strategy is not optimal. The presence of these two phases is the result of a competition between entropy and energy: by staying close to the interface the copolymer loses entropy, but it gains energy because it can more easily switch between the two sides of the interface in an attempt to place as many monomers as possible in their preferred solvent.

General monotonicity and convexity arguments show that $\mathcal{D}^{\text{que}}$ and $\mathcal{L}^{\text{que}}$ are separated by a *quenched critical curve* $\beta \mapsto h_c^{\text{que}}(\beta)$ given by

$$\begin{aligned} h_c^{\text{que}}(\beta) &= \sup\{h \geq 0 : g^{\text{que}}(\beta, h) > 0\} \\ &= \inf\{h \geq 0 : g^{\text{que}}(\beta, h) = 0\}, \qquad \beta \geq 0 \end{aligned} \tag{1.12}$$

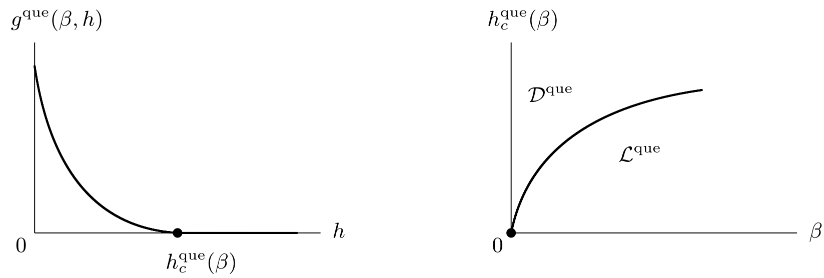

FIG. 2. *Qualitative pictures of* $h \mapsto g^{\mathrm{que}}(\beta, h)$ *for fixed* $\beta > 0$, *respectively,* $\beta \mapsto h_c^{\mathrm{que}}(\beta)$. *The quenched critical curve is part of* $\mathcal{D}^{\mathrm{que}}$.

with the property that $h_c^{\mathrm{que}}(0) = 0$, $\beta \mapsto h_c^{\mathrm{que}}(\beta)$ is nondecreasing and finite on $[0, \infty)$, and $\beta \mapsto \beta h_c^{\mathrm{que}}(\beta)$ is convex on $[0, \infty)$. Moreover, it is easy to check that $\lim_{\beta\to\infty} h_c^{\mathrm{que}}(\beta) = \sup[\mathrm{supp}(\nu)]$, the supremum of the support of ν (see Figure 2).

The following bounds are known for the quenched critical curve:

$$(1.13)\qquad \left(\frac{2\beta}{\alpha}\right)^{-1} M\left(\frac{2\beta}{\alpha}\right) \le h_c^{\mathrm{que}}(\beta) \le (2\beta)^{-1} M(2\beta) \qquad \forall \beta > 0.$$

The upper bound was proved in Bolthausen and den Hollander [9], and comes from an annealed estimate on ω. The lower bound was proved in Bodineau and Giacomin [7], and comes from strategies where the copolymer dips below the interface during rare stretches in ω where the empirical density is sufficiently biased downward. Since $M(\gamma) \sim \frac{1}{2}\gamma^2$ as $\gamma \to 0$ by (1.1), an immediate consequence of (1.13) is that $\beta \mapsto h_c^{\mathrm{que}}(\beta)$ is *strictly* increasing on $[0, \infty)$.

REMARK. In the literature, ρ is typically assumed to be *regularly varying at infinity*, that is,

$$(1.14)\qquad \rho(m) = m^{-\alpha} L(m) \quad \text{for some } \alpha \ge 1 \text{ with } L \text{ slowly varying at infinity.}$$

However, the proof of (1.13) in [9] and [7] can be extended to ρ satisfying the *much weaker* conditions in (1.6)–(1.7). In the literature, ν is sometimes assumed to have Gaussian or sub-Gaussian tails, which is stronger than (1.2). Also, this is not necessary for (1.13). Throughout our paper, (1.2) and (1.6)–(1.7) *are the only conditions in force* (with a sole exception indicated later on).

1.3. *Annealed free energy and critical curve.* Recalling (1.3)–(1.5), (1.8) and (1.10), and using that $\beta \sum_{k=1}^n (\omega_k + h) = \beta h n[1 + o(1)]$ ω-a.s. as $n \to \infty$, we see that the quenched excess free energy is given by

$$(1.15)\qquad g^{\mathrm{que}}(\beta, h) = \lim_{n\to\infty} \frac{1}{n} \log \widetilde{Z}_n^{\beta,h,\omega} \qquad \omega\text{-a.s.}$$

with

$$\widetilde{Z}_n^{\beta,h,\omega} = \int_\Pi P(d\pi) \exp\left[\beta \sum_{k=1}^{n} (\omega_k + h)[\mathrm{sign}(\pi_{k-1}, \pi_k) - 1]\right]. \tag{1.16}$$

In this partition sum, only the excursions of the copolymer below the interface contribute. The annealed version of the model has partition sum

$$\begin{aligned} &\mathbb{E}(\widetilde{Z}_n^{\beta,h,\omega}) \\ &\quad = \int_\Pi P(d\pi) \prod_{k=1}^{n} [1_{\{\mathrm{sign}(\pi_{k-1},\pi_k)=1\}} + e^{M(2\beta)-2\beta h} 1_{\{\mathrm{sign}(\pi_{k-1},\pi_k)=-1\}}], \end{aligned} \tag{1.17}$$

where $\mathbb{E}$ is expectation w.r.t. $\mathbb{P}$. The *annealed excess free energy* is therefore given by

$$g^{\mathrm{ann}}(\beta, h) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}(\widetilde{Z}_n^{\beta,h,\omega}). \tag{1.18}$$

[*Note*: In the annealed model, the average w.r.t. $\mathbb{P}$ is taken on the partition sum $\widetilde{Z}_n^{\beta,h,\omega}$ in (1.16) rather than on the original partition sum $Z_n^{\beta,h,\omega}$ in (1.5). If one defines $f^{\mathrm{ann}}(\beta, h)$ as in (1.18) using the expectation of $Z_n^{\beta,h,\omega}$, then $f^{\mathrm{ann}}(\beta, h) > \beta h$ whenever $\beta > 0$.] The two corresponding phases are

$$\begin{aligned} \mathcal{D}^{\mathrm{ann}} &= \{(\beta, h) : g^{\mathrm{ann}}(\beta, h) = 0\}, \\ \mathcal{L}^{\mathrm{ann}} &= \{(\beta, h) : g^{\mathrm{ann}}(\beta, h) > 0\}, \end{aligned} \tag{1.19}$$

which are referred to as the *annealed delocalized phase*, respectively, the *annealed localized phase*, and are separated by an *annealed critical curve* $\beta \mapsto h_c^{\mathrm{ann}}(\beta)$ given by

$$\begin{aligned} h_c^{\mathrm{ann}}(\beta) &= \sup\{h \geq 0 : g^{\mathrm{ann}}(\beta, h) > 0\} \\ &= \inf\{h \geq 0 : g^{\mathrm{ann}}(\beta, h) = 0\}, \qquad \beta \geq 0. \end{aligned} \tag{1.20}$$

An easy computation based on (1.17) gives that (see Figure 3)

$$g^{\mathrm{ann}}(\beta, h) = 0 \vee [M(2\beta) - 2\beta h], \qquad \beta, h \geq 0 \tag{1.21}$$

and

$$h_c^{\mathrm{ann}}(\beta) = (2\beta)^{-1} M(2\beta), \qquad \beta > 0. \tag{1.22}$$

Thus, the upper bound in (1.13) equals $h_c^{\mathrm{ann}}(\beta)$, while the lower bound equals $h_c^{\mathrm{ann}}(\beta/\alpha)$.

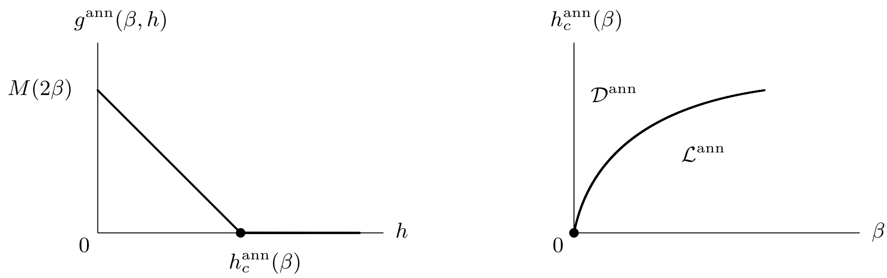

FIG. 3. *Qualitative picture of* $h \mapsto g^{\mathrm{ann}}(\beta,h)$ *for fixed* $\beta > 0$, *respectively*, $\beta \mapsto h_c^{\mathrm{ann}}(\beta)$. *The annealed critical curve is part of* $\mathcal{D}^{\mathrm{ann}}$.

1.4. *Main results*. Our variational characterization of the excess free energies and the critical curves are contained in the following theorem. Note that the case $h = 0$ is not included.

THEOREM 1.1. *Assume* (1.2) *and* (1.6)–(1.7).

(i) *For every* $\beta, h > 0$, *there are lower semicontinuous*, *convex*, *nonincreasing and possibly infinite functions*

$$g \mapsto S^{\mathrm{que}}(\beta,h;g), \qquad g \mapsto S^{\mathrm{ann}}(\beta,h;g), \tag{1.23}$$

satisfying $S^{\mathrm{que}} \le S^{\mathrm{ann}}$ *and given by explicit variational formulas*, *such that*

$$g^{\mathrm{que}}(\beta,h) = \inf\{g \in \mathbb{R} : S^{\mathrm{que}}(\beta,h;g) < 0\}, \qquad g^{\mathrm{ann}}(\beta,h) = \inf\{g \in \mathbb{R} : S^{\mathrm{ann}}(\beta,h;g) < 0\}. \tag{1.24}$$

(ii) *For every* $\beta > 0$, $g^{\mathrm{que}}(\beta,h)$ *and* $g^{\mathrm{ann}}(\beta,h)$ *are the unique solutions of the equations*

$$S^{\mathrm{que}}(\beta,h;g) = 0 \quad \textit{for } 0 < h \le h_c^{\mathrm{que}}(\beta), \qquad S^{\mathrm{ann}}(\beta,h;g) = 0 \quad \textit{for } h = h_c^{\mathrm{ann}}(\beta). \tag{1.25}$$

(iii) *For every* $\beta > 0$, $h_c^{\mathrm{que}}(\beta)$ *and* $h_c^{\mathrm{ann}}(\beta)$ *are the unique solutions of the equations*

$$S^{\mathrm{que}}(\beta,h;0) = 0, \qquad S^{\mathrm{ann}}(\beta,h;0) = 0. \tag{1.26}$$

The variational formulas for $S^{\mathrm{que}}(\beta,h;g)$ and $S^{\mathrm{ann}}(\beta,h;g)$ are given in Theorem 3.1, respectively, Theorem 3.2 in Section 3. Figures 6–9 in Section 3 show how these functions depend on β, h and g, which is crucial for our analysis.

We state seven corollaries that are consequences of the variational formulas. The first three corollaries are strict inequalities for the excess free energies and the critical curves.

COROLLARY 1.2. $g^{\mathrm{que}}(\beta,h) < g^{\mathrm{ann}}(\beta,h)$ *for all* $(\beta,h) \in \mathcal{L}^{\mathrm{ann}}$.

COROLLARY 1.3. *If* $\alpha > 1$, *then* $h_c^{\mathrm{que}}(\beta) < h_c^{\mathrm{ann}}(\beta)$ *for all* $\beta > 0$.

COROLLARY 1.4. *If* $\alpha > 1$, *then* $h_c^{\mathrm{que}}(\beta) > h_c^{\mathrm{ann}}(\beta/\alpha)$ *for all* $\beta > 0$.

Note that $h_c^{\mathrm{que}}(\beta) = h_c^{\mathrm{ann}}(\beta)$ for all $\beta > 0$ when $\alpha = 1$, by (1.13).

The next two corollaries concern the slope of the quenched critical curve at $\beta = 0$. Abbreviate $m_\rho = \sum_{n\in\mathbb{N}} n\rho(n)$. We say that ρ is *standard* when it is asymptotically periodic [i.e., supp(ρ) eventually coincides with $p\mathbb{N}$ for some $p \in \mathbb{N}$] and is regularly varying at infinity [i.e., (1.14) holds along supp(ρ)].

COROLLARY 1.5. *Suppose that either* $m_\rho < \infty$ *or* ρ *is standard with* $m_\rho = \infty$ *and* $\alpha = 2$. *Then* $\liminf_{\beta\downarrow 0} h_c^{\mathrm{que}}(\beta)/\beta \geq K_c^*(\alpha)$ *with* $K_c^*(\alpha) = \frac{1+\alpha}{2\alpha}$ (*see Figure* 4).

For $1 < \alpha < 2$, let

$$I_\alpha(B) = \int_0^\infty dy\, y^{-\alpha}\big[E_\alpha(y,B) - 1\big], \qquad B \geq 1, \tag{1.27}$$

where

$$E_\alpha(y,B) = \int_{\mathbb{R}} dx \frac{1}{\sqrt{2\pi}} e^{-(1/2)x^2} f_\alpha\big(e^{-2By-2\sqrt{y}x}\big) \tag{1.28}$$

with

$$f_\alpha(z) = \big\{\tfrac{1}{2}\big(1+z^\alpha\big)\big\}^{1/\alpha} \tag{1.29}$$

and let $1 < B(\alpha) < \infty$ be the unique solution of the equation $I_\alpha(B) = 0$.

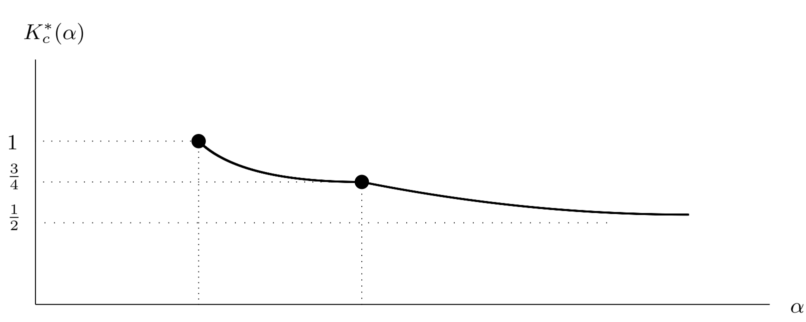

FIG. 4. *Qualitative picture of* $\alpha \mapsto K_c^*(\alpha)$.

COROLLARY 1.6. *Suppose that ρ is standard with $m_\rho = \infty$ and $1 < \alpha < 2$. Then* $\liminf_{\beta \downarrow 0} h_c^{\mathrm{que}}(\beta)/\beta \geq K_c^*(\alpha)$ *with* $K_c^*(\alpha) = \frac{B(\alpha)}{\alpha}$ (*see Figure* 4).

The last two corollaries concern the typical path behavior. Let $\widetilde{\mathcal{P}}_n^{\beta,h,\omega}$ denote the path measure associated with the constrained partition sum $\widetilde{Z}_n^{\beta,h,\omega}$ defined in (1.16). Write $\mathcal{M}_n = |\{1 \leq i \leq n : \pi_i = 0\}|$ to denote the number of times π returns to the interface up to time n. Define

$$\overline{\mathcal{D}}^{\mathrm{que}} = \{(\beta, h) : \overline{S}^{\mathrm{que}}(\beta, h; 0) \leq 0\}, \tag{1.30}$$

where $\overline{S}^{\mathrm{que}}(\beta, h; g)$, for $g \in [0, \infty)$, is defined in (3.16) below.

COROLLARY 1.7. *For every $(\beta, h) \in \mathrm{int}(\overline{\mathcal{D}}^{\mathrm{que}})$ and $c > \alpha/[-\overline{S}^{\mathrm{que}}(\beta, h; 0)] \in (0, \infty)$,*

$$\lim_{n\to\infty} \widetilde{\mathcal{P}}_n^{\beta,h,\omega}(\mathcal{M}_n \geq c \log n) = 0 \qquad \omega\text{-}a.s. \tag{1.31}$$

COROLLARY 1.8. *For every $(\beta, h) \in \mathcal{L}^{\mathrm{que}}$,*

$$\lim_{n\to\infty} \widetilde{\mathcal{P}}_n^{\beta,h,\omega}\left(\left|\frac{1}{n}\mathcal{M}_n - C\right| \leq \varepsilon\right) = 1 \qquad \omega\text{-}a.s.\ \forall \varepsilon > 0, \tag{1.32}$$

where

$$-\frac{1}{C} = \frac{\partial}{\partial g} S^{\mathrm{que}}(\beta, h; g^{\mathrm{que}}(\beta, h)) \in (-\infty, 0), \tag{1.33}$$

provided this derivative exists. (*By convexity, at least the left-derivative and the right-derivative exist*.)

1.5. *Discussion*. 1. The main importance of our results in Section 1.4 is that they open up a window on the copolymer model with a variational view. Whereas the results in the literature were obtained with the help of a variety of *estimation techniques*, Theorem 1.1 provides *variational formulas* that are new and explicit. As we will see in Section 3, these variational formulas are not easy to manipulate. However, they provide a natural setting, and are robust in the sense that the large deviation principles on which they are based (see Section 2) can be applied to other polymer models as well, for example, the pinning model with disorder (Cheliotis and den Hollander [14]). Still other applications involve certain classes of interacting stochastic systems (Birkner, Greven and den Hollander [5]). For an overview, see den Hollander [17].

2. The gap between the excess free energies stated in Corollary 1.2 has never been claimed in the literature, but follows from known results. Fix $\beta > 0$. We know that $h \mapsto g^{\mathrm{ann}}(\beta, h)$ is strictly positive, strictly decreasing and linear on $(0, h_c^{\mathrm{ann}}(\beta)]$, and zero on $[h_c^{\mathrm{ann}}(\beta), \infty)$ (see Figure 3). We also know that $h \mapsto g^{\mathrm{que}}(\beta, h)$ is strictly positive, strictly decreasing and convex on $(0, h_c^{\mathrm{que}}(\beta)]$, and

zero on $[h_c^{\mathrm{que}}(\beta), \infty)$. It was shown in Giacomin and Toninelli [23, 24] that $h \mapsto g^{\mathrm{que}}(\beta, h)$ drops below a quadratic as $h \uparrow h_c^{\mathrm{que}}(\beta)$, that is, the phase transition is "at least of second order" (see Figure 2). Hence, the gap is present in a left-neighborhood of $h_c^{\mathrm{que}}(\beta)$. Combining this observation with the fact that $g^{\mathrm{que}}(\beta, h) \le g^{\mathrm{ann}}(\beta, h)$ and $h_c^{\mathrm{que}}(\beta) \le h_c^{\mathrm{ann}}(\beta)$, it follows that the gap is present for all $h \in (0, h_c^{\mathrm{ann}}(\beta))$. *Note*: The above argument crucially relies on the linearity of $h \mapsto g^{\mathrm{ann}}(\beta, h)$ on $(0, h_c^{\mathrm{ann}}(\beta)]$. However, we will see in Section 4 that our proof of Corollary 1.2 is robust and does not depend on this linearity.

3. For a number of years, all attempts in the literature to improve (1.13) had failed. As explained in Orlandini, Rechnitzer and Whittington [27] and Caravenna and Giacomin [10], the reason behind this failure is that any improvement of (1.13) necessarily requires a deep understanding of the global behavior of the copolymer when the parameters are close to the quenched critical curve. Toninelli [28] proved the strict upper bound in Corollary 1.3 with the help of *fractional moment estimates* for unbounded disorder and large β subject to (1.2) and (1.14), and this result was later extended by Bodineau, Giacomin, Lacoin and Toninelli [8] to arbitrary disorder and arbitrary β, again subject to (1.2) and (1.14). The latter paper also proved the strict lower bound in Corollary 1.4 with the help of *appropriate localization strategies* for small β and $\alpha \ge \alpha_0$, where $\alpha_0 \approx 1.801$ (theoretical bound) and $\alpha_0 \approx 1.65$ (numerical bound), which unfortunately excludes the simple random walk example in Section 1.1 for which $\alpha = \frac{3}{2}$. Corollaries 1.3 and 1.4 settle the strict inequalities in full generality subject to (1.2) and (1.6)–(1.7).

4. A point of heated debate has been the value of

$$K_c = \lim_{\beta \downarrow 0} h_c^{\mathrm{que}}(\beta)/\beta, \tag{1.34}$$

which is believed to be *universal*, that is, to depend on α alone and to be robust under changes of the fine details of the interaction Hamiltonian. The *existence* of K_c was proved in Bolthausen and den Hollander [9] for ρ associated with simple random walk ($\alpha = \frac{3}{2}$) and for binary disorder (the proof uses a Brownian approximation of the copolymer model). This result was extended in Caravenna and Giacomin [11] to ρ satisfying (1.14) with $1 < \alpha < 2$ and to disorder with a moment generating function that is finite in a neighborhood of the origin (the proof uses a Lévy approximation of the copolymer model). No value for K_c was identified. For $\alpha \ge 2$, even the existence of K_c remained open. The bounds in (1.13) imply that $K_c \in [1/\alpha, 1]$, and various claims were made in the literature arguing in favor of $K_c = 1/\alpha$, respectively, $K_c = 1$. However, in Bodineau, Giacomin, Lacoin and Toninelli [8] it was shown that $\liminf_{\beta \downarrow 0} h_c^{\mathrm{que}}(\beta)/\beta > 1/\alpha$ for $\alpha \ge \alpha_0$ and $\liminf_{\beta \downarrow 0} h_c^{\mathrm{que}}(\beta)/\beta \ge \frac{1}{2} \vee (1/\sqrt{\alpha})$ for $\alpha > 2$. Corollaries 1.5 and 1.6 improve these results. We do not have an upper bound, but conjecture that for $\alpha \ge 2$ it coincides with our lower bound[5]. In [8] it was shown that $\limsup_{\beta\downarrow 0} h_c^{\mathrm{que}}(\beta)/\beta < 1$ for $\alpha > 2$, which was later extended to $\alpha > 1$ in Toninelli [29]. For an overview, see Caravenna, Giacomin and Toninelli [13].

5. A numerical analysis for simple random walk ($\alpha = \frac{3}{2}$) and binary disorder carried out in Caravenna, Giacomin and Gubinelli [12] (see also Giacomin [21], Chapter 9) showed that $K_c \in [0.82, 0.84]$. Since $\frac{5}{6} = 0.833\ldots$, it is natural to wonder whether $K_c = \frac{1+\alpha}{2\alpha}$ for all $\alpha > 1$. In [12], it was also shown for simple random walk and binary disorder that

$$h_c^{\mathrm{que}}(\beta) \approx (2K_c\beta)^{-1} \log\cosh(2K_c\beta) \qquad \text{for moderate } \beta. \tag{1.35}$$

Thus, the quenched critical curve lies "somewhere halfway" between the two bounds in (1.13), and so it remains a challenge to quantify the strict inequalities in Corollaries 1.3 and 1.4. Some quantification for the upper bound was offered in Bodineau, Giacomin, Lacoin and Toninelli [8], and for the lower bound in Toninelli [29]. Our proofs of Corollaries 1.3 and 1.4 sharpen these quantifications.

6. Because of (1.13), it was suggested that the quenched critical curve possibly depends on the exponent α of ρ alone and not on the fine details of ρ. However, it was shown in Bodineau, Giacomin, Lacoin and Toninelli [8] that, subject to (1.2), for every $\alpha > 1$, $\beta > 0$ and $\epsilon > 0$ there exists a ρ satisfying (1.14) such that $h_c^{\mathrm{que}}(\beta)$ is ϵ-close to the upper bound, which rules out such a scenario. Our variational characterization in Section 3 confirms this observation, and makes it quite evident that the fine details of ρ do indeed matter.

7. Special cases of Corollaries 1.7 and 1.8 were proved in Biskup and den Hollander [6], for simple random walk and binary disorder, and in Giacomin and Toninelli [22, 25], subject to (1.14) and for disorder satisfying a Gaussian concentration of measure bound. (Actually, the latter two papers deal with the time spent below the interface.) However, no formulas were obtained for the relevant constants. The latter two papers prove the bound under the average quenched measure, that is, under $\mathbb{E}(P_n^{\beta,h,\omega})$. For the pinning model with disorder, the same result as in Corollary 1.7 was derived in Mourrat [26] with the help of the variational characterization obtained in Cheliotis and den Hollander [14].

8. We will see in Section 3.1 that the region $\overline{\mathcal{D}}^{\mathrm{que}}$ defined in (1.30) and used in Corollary 1.7 is contained in the quenched delocalized region $\mathcal{D}^{\mathrm{que}}$. The two regions coincide when

$$\lim_{g\downarrow 0} \overline{S}^{\mathrm{que}}(\beta, h; g) = \overline{S}^{\mathrm{que}}(\beta, h; 0). \tag{1.36}$$

The latter condition holds for the pinning model (den Hollander and Opoku [18]). We believe it also holds for the copolymer model, but we are unable to prove this.

[5] This conjecture was taken up and proved by Berger, Caravenna, Poisat, Sun and Zygouras [1].

1.6. *Outline*. In Section 2, we recall two large deviation principles (LDP's) derived in Birkner [3] and Birkner, Greven and den Hollander [4], which describe the large deviation behavior of the empirical process of words cut out from a random letter sequence according to a random renewal process with exponentially bounded, respectively, polynomial tails. In Section 3, we use these LDPs to prove Theorem 1.1. In Sections 4–8, we prove Corollaries 1.2–1.8. Appendices A–D contain a number of technical estimates that are needed in Section 3.

In Cheliotis and den Hollander [14], the LDPs in [4] were applied to the pinning model with disorder, and variational formulas were derived for the critical curves (not the free energies). The Hamiltonian is similar in spirit to (1.4), except that the disorder is felt only *at* the interface, which makes the pinning model easier than the copolymer model. The present paper borrows ideas from [14]. However, the new challenges that come up are considerable.

2. Large deviation principles: intermezzo. In this section, we recall the LDPs from Birkner [3] and Birkner, Greven and den Hollander [4], which are the key tools in the present paper. Section 2.1 introduces the relevant notation, while Sections 2.2 and 2.3 state the annealed, respectively, quenched version of the LDP. Apart from minor modifications, this section is copied from [4]. We repeat it here in order to set the notation and to keep the paper self-contained.

2.1. *Notation*. Let E be a Polish space, playing the role of an alphabet, that is, a set of *letters*. Let $\widetilde{E} = \bigcup_{k\in\mathbb{N}} E^k$ be the set of *finite words* drawn from E, which can be metrized to become a Polish space. Write $\mathcal{P}(E)$ and $\mathcal{P}(\widetilde{E})$ to denote the set of probability measures on E and $\widetilde{E}$ (endowed with the topology of weak convergence).

Fix $\nu \in \mathcal{P}(E)$, and $\rho \in \mathcal{P}(\mathbb{N})$ satisfying (1.6). Let $X = (X_k)_{k\in\mathbb{N}}$ be i.i.d. E-valued random variables with marginal law ν, and $\tau = (\tau_i)_{i\in\mathbb{N}}$ i.i.d. $\mathbb{N}$-valued random variables with marginal law ρ. Assume that X and τ are independent, and write $\mathbb{P}^* = \mathbb{P} \otimes P^*$ to denote their joint law. Cut words out of the letter sequence X according to τ (see Figure 5), that is, put

$$T_0 = 0 \quad \text{and} \quad T_i = T_{i-1} + \tau_i, \qquad i \in \mathbb{N} \tag{2.1}$$

and let

$$Y^{(i)} = (X_{T_{i-1}+1}, X_{T_{i-1}+2}, \dots, X_{T_i}), \qquad i \in \mathbb{N}. \tag{2.2}$$

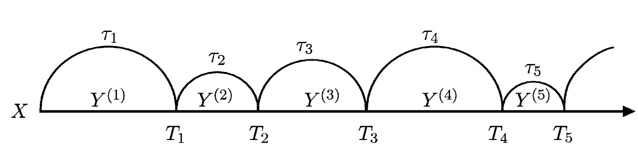

FIG. 5. *Cutting words out from a sequence of letters according to renewal times.*

Under the law $\mathbb{P}^*$, $Y = (Y^{(i)})_{i\in\mathbb{N}}$ is an i.i.d. sequence of words with marginal law $q_{\rho,\nu}$ on $\widetilde{E}$ given by

$$
\begin{aligned}
q_{\rho,\nu}(dx_1,\dots,dx_m) &= \mathbb{P}^*\big(Y^{(1)} \in (dx_1,\dots,dx_m)\big)\\
&= \rho(m)\nu(dx_1)\times\cdots\times\nu(dx_m),\\
&\qquad m\in\mathbb{N}, x_1,\dots,x_m\in E.
\end{aligned}
\tag{2.3}
$$

We define ρ_g as the tilted version of ρ given by

$$
\begin{aligned}
\rho_g(m) &= \frac{e^{-gm}\rho(m)}{\mathcal{N}(g)}, \qquad m\in\mathbb{N},\\
\mathcal{N}(g) &= \sum_{m\in\mathbb{N}} e^{-gm}\rho(m), \qquad g\in[0,\infty).
\end{aligned}
\tag{2.4}
$$

Note that if $g>0$, then ρ_g has an *exponentially bounded tail*. For $g=0$ we write ρ instead of ρ_0. We write P_g^* and $q_{\rho_g,\nu}$ for the analogues of P^* and $q_{\rho,\nu}$ when ρ is replaced by ρ_g defined in (2.4).

The reverse operation of *cutting* words out of a sequence of letters is *gluing* words together into a sequence of letters. Formally, this is done by defining a *concatenation* map κ from $\widetilde{E}^{\mathbb{N}}$ to $E^{\mathbb{N}}$. This map induces in a natural way a map from $\mathcal{P}(\widetilde{E}^{\mathbb{N}})$ to $\mathcal{P}(E^{\mathbb{N}})$, the sets of probability measures on $\widetilde{E}^{\mathbb{N}}$ and $E^{\mathbb{N}}$ (endowed with the topology of weak convergence). The concatenation $q_{\rho,\nu}^{\otimes\mathbb{N}}\circ\kappa^{-1}$ of $q_{\rho,\nu}^{\otimes\mathbb{N}}$ equals $\nu^{\mathbb{N}}$, as is evident from (2.3).

Let $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ be the set of probability measures on $\widetilde{E}^{\mathbb{N}}$ that are invariant under the left-shift $\tilde{\theta}$ acting on $\widetilde{E}^{\mathbb{N}}$. For $Q\in\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$, let $H(Q|q_{\rho,\nu}^{\otimes\mathbb{N}})$ be the *specific relative entropy of Q w.r.t. $q_{\rho,\nu}^{\otimes\mathbb{N}}$* defined by

$$
H\big(Q|q_{\rho,\nu}^{\otimes\mathbb{N}}\big) = \lim_{N\to\infty}\frac{1}{N}h\big(\tilde{\pi}_N Q|q_{\rho,\nu}^{N}\big),
\tag{2.5}
$$

where $\tilde{\pi}_N Q\in\mathcal{P}(\widetilde{E}^N)$ denotes the projection of Q onto the first N words, $h(\cdot|\cdot)$ denotes relative entropy, and the limit is nondecreasing. The following lemma relates the specific relative entropies of Q w.r.t. $q_{\rho,\nu}^{\otimes\mathbb{N}}$ and $q_{\rho_g,\nu}^{\otimes\mathbb{N}}$.

LEMMA 2.1. *For $Q\in\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ and $g\in[0,\infty)$,*

$$
H\big(Q|q_{\rho_g,\nu}^{\otimes\mathbb{N}}\big) = H\big(Q|q_{\rho,\nu}^{\otimes\mathbb{N}}\big) + \log\mathcal{N}(g) + g m_Q
\tag{2.6}
$$

with $\mathcal{N}(g)\in(0,1]$ defined in (2.4) *and $m_Q = E_Q(\tau_1)\in[1,\infty]$ the average word length under Q (E_Q denotes expectation under the law Q and τ_1 is the length of the first word*).

PROOF. Observe from (2.4) that

$$
\begin{aligned}
h(\tilde{\pi}_N Q|q^N_{\rho_g,\nu}) &= \int_{\widetilde{E}^N} (\tilde{\pi}_N Q)(dy) \log\left(\frac{d\tilde{\pi}_N Q}{dq^N_{\rho_g,\nu}}(y)\right) \\
&= \int_{\widetilde{E}^N} (\tilde{\pi}_N Q)(dy) \log\left(\frac{\mathcal{N}(g)^N}{e^{-g\sum_{i=1}^N |y^{(i)}|}} \frac{d\tilde{\pi}_N Q}{dq^N_{\rho,\nu}}(y)\right) \qquad (2.7)\\
&= h(\tilde{\pi}_N Q|q^N_{\rho,\nu}) + N \log \mathcal{N}(g) + N g m_Q,
\end{aligned}
$$

where $|y^{(i)}|$ is the length of the ith word and the second equality uses that $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$. Let $N \to \infty$ and use (2.5), to get the claim. □

Lemma 2.1 implies that if $g > 0$, then $m_Q < \infty$ whenever $H(Q|q^{\otimes\mathbb{N}}_{\rho_g,\nu}) < \infty$. This is a special case of [3], Lemma 7.

2.2. *Annealed LDP.* For $N \in \mathbb{N}$, let $(Y^{(1)}, \dots, Y^{(N)})^{\mathrm{per}}$ be the periodic extension of the N-tuple $(Y^{(1)}, \dots, Y^{(N)}) \in \widetilde{E}^N$ to an element of $\widetilde{E}^{\mathbb{N}}$, and define

$$
R_N^X = \frac{1}{N} \sum_{i=0}^{N-1} \delta_{\tilde{\theta}^i (Y^{(1)},\dots,Y^{(N)})^{\mathrm{per}}} \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}). \qquad (2.8)
$$

This is the *empirical process of N-tuples of words*. The superscript X indicates that the words $Y^{(1)}, \dots, Y^{(N)}$ are cut from the letter sequence X. The following *annealed LDP* is standard (see, e.g., Dembo and Zeitouni [15], Section 6.5).

THEOREM 2.2. *For every $g \in [0, \infty)$, the family $(\mathbb{P} \times P_g^*)(R_N^X \in \cdot)$, $N \in \mathbb{N}$, satisfies the LDP on $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ with rate N and with rate function I_g^{ann} given by*

$$
I_g^{\mathrm{ann}}(Q) = H(Q|q^{\otimes\mathbb{N}}_{\rho_g,\nu}), \qquad Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}). \qquad (2.9)
$$

This rate function is lower semicontinuous, has compact level sets, has a unique zero at $q^{\otimes\mathbb{N}}_{\rho_g,\nu}$, and is affine.

It follows from Lemma 2.1 that

$$
I_g^{\mathrm{ann}}(Q) = I^{\mathrm{ann}}(Q) + \log \mathcal{N}(g) + g m_Q, \qquad (2.10)
$$

where $I^{\mathrm{ann}}(Q) = H(Q|q^{\otimes\mathbb{N}}_{\rho,\nu})$, the annealed rate function for $g = 0$.

2.3. *Quenched LDP.* To formulate the quenched analogue of Theorem 2.2, we need some more notation. Let $\mathcal{P}^{\mathrm{inv}}(E^{\mathbb{N}})$ be the set of probability measures on $E^{\mathbb{N}}$ that are invariant under the left-shift θ acting on $E^{\mathbb{N}}$. For $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ such that $m_Q < \infty$, define

$$
\Psi_Q = \frac{1}{m_Q} E_Q\left(\sum_{k=0}^{\tau_1 - 1} \delta_{\theta^k \kappa(Y)}\right) \in \mathcal{P}^{\mathrm{inv}}(E^{\mathbb{N}}). \qquad (2.11)
$$

Think of Ψ_Q as the shift-invariant version of $Q \circ \kappa^{-1}$ obtained after *randomizing* the location of the origin. This randomization is necessary because a shift-invariant Q in general does not give rise to a shift-invariant $Q \circ \kappa^{-1}$.

For $\mathrm{tr} \in \mathbb{N}$, let $[\cdot]_{\mathrm{tr}} : \widetilde{E} \to [\widetilde{E}]_{\mathrm{tr}} = \bigcup_{k=1}^{\mathrm{tr}} E^k$ denote the *truncation map* on words defined by

$$y = (x_1, \dots, x_m) \mapsto [y]_{\mathrm{tr}} = (x_1, \dots, x_{m \wedge \mathrm{tr}}), \qquad m \in \mathbb{N}, x_1, \dots, x_m \in E, \tag{2.12}$$

that is, $[y]_{\mathrm{tr}}$ is the word of length $\le \mathrm{tr}$ obtained from the word y by dropping all the letters with label $> \mathrm{tr}$. This map induces in a natural way a map from $\widetilde{E}^{\mathbb{N}}$ to $[\widetilde{E}]_{\mathrm{tr}}^{\mathbb{N}}$, and from $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ to $\mathcal{P}^{\mathrm{inv}}([\widetilde{E}]_{\mathrm{tr}}^{\mathbb{N}})$. Note that if $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$, then $[Q]_{\mathrm{tr}}$ is an element of the set

$$\mathcal{P}^{\mathrm{inv,fin}}(\widetilde{E}^{\mathbb{N}}) = \{Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}) : m_Q < \infty\}. \tag{2.13}$$

Define (w-lim means weak limit)

$$\mathcal{R} = \left\{ Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}) : \operatorname*{w-lim}_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \delta_{\theta^k \kappa(Y)} = \nu^{\otimes \mathbb{N}} \ Q\text{-a.s.} \right\}, \tag{2.14}$$

that is, the set of probability measures in $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ under which the concatenation of words almost surely has the same asymptotic statistics as a typical realization of X.

THEOREM 2.3 (Birkner [3]; Birkner, Greven and den Hollander [4]). *Assume* (1.2) *and* (1.6). *Then, for $\nu^{\otimes\mathbb{N}}$-a.s. all X and all $g \in [0, \infty)$, the family of* (*regular*) *conditional probability distributions $P_g^*(R_N^X \in \cdot)$, $N \in \mathbb{N}$, satisfies the LDP on $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ with rate N and with deterministic rate function I_g^{que} given by*

$$I_g^{\mathrm{que}}(Q) = \begin{cases} I_g^{\mathrm{ann}}(Q), & \text{if } Q \in \mathcal{R}, \\ \infty, & \text{otherwise}, \end{cases} \qquad \text{when } g > 0 \tag{2.15}$$

and

$$I^{\mathrm{que}}(Q) = \begin{cases} I^{\mathrm{fin}}(Q), & \text{if } Q \in \mathcal{P}^{\mathrm{inv,fin}}(\widetilde{E}^{\mathbb{N}}), \\ \lim_{\mathrm{tr} \to \infty} I^{\mathrm{fin}}([Q]_{\mathrm{tr}}), & \text{otherwise}, \end{cases} \qquad \text{when } g = 0, \tag{2.16}$$

where

$$I^{\mathrm{fin}}(Q) = H(Q | q_{\rho,\nu}^{\otimes\mathbb{N}}) + (\alpha - 1) m_Q H(\Psi_Q | \nu^{\otimes\mathbb{N}}). \tag{2.17}$$

This rate function is lower semicontinuous, has compact level sets, has a unique zero at $q_{\rho_g,\nu}^{\otimes\mathbb{N}}$, and is affine.

The difference between (2.15) for $g > 0$ and (2.16)–(2.17) for $g = 0$ can be explained as follows. For $g = 0$, the word length distribution ρ has a polynomial tail. It therefore is only exponentially costly to cut out a few words of an exponentially large length in order to move to stretches in X that are suitable to build a large deviation $\{R_N^X \approx Q\}$ with words whose length is of order 1. This is precisely where the second term in (2.17) comes from: this term is the extra cost to find these stretches under the quenched law rather than to create them "on the spot" under the annealed law. For $g > 0$, on the other hand, the word length distribution ρ_g has an exponentially bounded tail, and hence exponentially long words are too costly, so that suitable stretches far away cannot be reached. Phrased differently, $g > 0$ and $\alpha \in [1, \infty)$ is qualitatively similar to $g = 0$ and $\alpha = \infty$, for which we see that the expression in (2.17) is finite if and only $\Psi_Q = \nu^{\otimes\mathbb{N}}$. It was shown in [3], Lemma 2, that

$$\Psi_Q = \nu^{\otimes\mathbb{N}} \quad \Longleftrightarrow \quad Q \in \mathcal{R} \qquad \text{on } \mathcal{P}^{\mathrm{inv,fin}}(\widetilde{E}^{\mathbb{N}}) \tag{2.18}$$

and so this explains why the restriction $Q \in \mathcal{R}$ appears in (2.15). For more background, see [4].

Note that $I^{\mathrm{que}}(Q)$ requires a truncation approximation when $m_Q = \infty$, for which case there is no closed form expression like in (2.17). As we will see later on, the cases $m_Q < \infty$ and $m_Q = \infty$ need to be separated. It was shown in [4] that for all $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$,

$$\begin{aligned} I^{\mathrm{ann}}(Q) &= \lim_{\mathrm{tr}\to\infty} I^{\mathrm{ann}}([Q]_{\mathrm{tr}}), \\ I^{\mathrm{que}}(Q) &= \lim_{\mathrm{tr}\to\infty} I^{\mathrm{que}}([Q]_{\mathrm{tr}}). \end{aligned} \tag{2.19}$$

3. Proof of Theorem 1.1. We are now ready to return to the copolymer and start our variational analysis.

In Sections 3.1 and 3.2, we derive the variational formulas for the quenched and the annealed excess free energies and critical curves that were announced in Theorem 1.1. These variational formulas are stated in Theorems 3.1 and 3.2 below and imply part (i) of Theorem 1.1. In Section 3.3, we state additional properties that imply parts (ii) and (iii).

3.1. *Quenched excess free energy and critical curve.* Let

$$\widetilde{Z}_{n,0}^{\beta,h,\omega} = E\left(\exp\left[\beta \sum_{k=1}^{n} (\omega_k + h)\left[\mathrm{sign}(\pi_{k-1}, \pi_k) - 1\right]\right] 1_{\{\pi_n = 0\}}\right), \tag{3.1}$$

which differs from $\widetilde{Z}_n^{\beta,h,\omega}$ in (1.16) because of the extra indicator $1_{\{\pi_n=0\}}$. This indicator is harmless in the limit as $n \to \infty$ (see Bolthausen and den Hollander [9], Lemma 2) and is added for convenience. To derive a variational expression for $g^{\mathrm{que}}(\beta,h)=\lim_{n\to\infty}\frac{1}{n}\log\widetilde{Z}_{n,0}^{\beta,h,\omega}$ ω-a.s., we use Theorem 2.3 with

$$X=\omega,\qquad E=\mathbb{R},\qquad \widetilde{E}=\bigcup_{k\in\mathbb{N}}\mathbb{R}^k,\qquad \nu\in\mathcal{P}(\mathbb{R}),\qquad \rho\in\mathcal{P}(\mathbb{N}),\tag{3.2}$$

where ν satisfies (1.2) and ρ satisfies (1.6)–(1.7), with $\rho(n)=P(\{\pi\in\Pi:\pi_k\neq 0\ \forall 1\le k<n,\pi_n=0\})$, $n\in\mathbb{N}$, the excursion length distribution.

Abbreviate

$$\mathcal{C}=\{Q\in\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}):I^{\mathrm{ann}}(Q)<\infty\},\qquad \mathcal{C}^{\mathrm{fin}}=\{Q\in\mathcal{C}:m_Q<\infty\}.\tag{3.3}$$

THEOREM 3.1. *Assume* (1.2) *and* (1.6)–(1.7). *Fix* $\beta,h>0$.

(i) *The quenched excess free energy is given by*

$$g^{\mathrm{que}}(\beta,h)=\inf\{g\in\mathbb{R}:S^{\mathrm{que}}(\beta,h;g)<0\},\tag{3.4}$$

where

$$S^{\mathrm{que}}(\beta,h;g)=\sup_{Q\in\mathcal{C}^{\mathrm{fin}}\cap\mathcal{R}}[\Phi_{\beta,h}(Q)-gm_Q-I^{\mathrm{ann}}(Q)]\tag{3.5}$$

with

$$\Phi_{\beta,h}(Q)=\int_{\widetilde{E}}(\tilde{\pi}_1Q)(dy)\log\phi_{\beta,h}(y),\tag{3.6}$$

$$\phi_{\beta,h}(y)=\tfrac{1}{2}\big(1+e^{-2\beta h\tau(y)-2\beta\sigma(y)}\big),\tag{3.7}$$

where $\tilde{\pi}_1:\widetilde{E}^{\mathbb{N}}\to\widetilde{E}$ *is the projection onto the first word, that is,* $\tilde{\pi}_1Q=Q\circ\tilde{\pi}_1^{-1}$, *and* $\tau(y),\sigma(y)$ *are the length, respectively, the sum of the letters in the word* y.

(ii) *An alternative variational formula at* $g=0$ *is* $S^{\mathrm{que}}(\beta,h;0)=S_*^{\mathrm{que}}(\beta,h)$ *with*

$$S_*^{\mathrm{que}}(\beta,h)=\sup_{Q\in\mathcal{C}^{\mathrm{fin}}}[\Phi_{\beta,h}(Q)-I^{\mathrm{que}}(Q)].\tag{3.8}$$

(iii) *The function* $g\mapsto S^{\mathrm{que}}(\beta,h;g)$ *is lower semicontinuous, convex and non-increasing on* $\mathbb{R}$, *is infinite on* $(-\infty,0)$, *and is finite, continuous and strictly decreasing on* $(0,\infty)$.

PROOF. The proof comes in 7 steps. Throughout the proof, $\beta,h>0$ are fixed.

1. Let $t_n=t_n(\pi)$ denote the number of excursions in π away from the interface [recall that $\pi_n=0$ in (3.1)]. For $i=1,\dots,t_n$, let $I_i=I_i(\pi)$ denote the ith excursion interval in π. Then

$$\begin{aligned}&\beta\sum_{k=1}^{n}(\omega_k+h)[\mathrm{sign}(\pi_{k-1},\pi_k)-1]\\&\quad=\beta\sum_{i=1}^{t_n}\sum_{k\in I_i}(\omega_k+h)[\mathrm{sign}(\pi_{k-1},\pi_k)-1].\end{aligned}\tag{3.9}$$

During the ith excursion, π cuts out the word $\omega_{I_i} = (\omega_k)_{k\in I_i}$ from ω. Each excursion can be either above or below the interface, with probability $\frac{1}{2}$ each, and so the contribution to $\widetilde{Z}_{n,0}^{\beta,h,\omega}$ in (3.1) coming from the ith excursion is

$$\psi^{\omega}_{\beta,h}(I_i) = \frac{1}{2}\left(1 + \exp\left[-2\beta \sum_{k\in I_i}(\omega_k + h)\right]\right). \tag{3.10}$$

Hence, putting $I_i = (k_{i-1}, k_i] \cap \mathbb{N}$, we have

$$\widetilde{Z}_{n,0}^{\beta,h,\omega} = \sum_{N\in\mathbb{N}} \sum_{0=k_0<k_1<\cdots<k_N=n} \prod_{i=1}^{N} \rho(k_i - k_{i-1})\psi^{\omega}_{\beta,h}\big((k_{i-1}, k_i]\big). \tag{3.11}$$

Summing on n, we get

$$\sum_{n\in\mathbb{N}} e^{-gn}\widetilde{Z}_{n,0}^{\beta,h,\omega} = \sum_{N\in\mathbb{N}} F_N^{\beta,h,\omega}(g), \qquad g \in [0,\infty) \tag{3.12}$$

with [recall (2.4)]

$$\begin{aligned} F_N^{\beta,h,\omega}(g) = \mathcal{N}(g)^N &\sum_{0=k_0<k_1<\cdots<k_N<\infty} \left(\prod_{i=1}^{N} \rho_g(k_i - k_{i-1})\right) \\ &\times \exp\left[\sum_{i=1}^{N} \log \psi^{\omega}_{\beta,h}\big((k_{i-1}, k_i]\big)\right], \\ &g \in [0,\infty). \end{aligned} \tag{3.13}$$

2. Let

$$R_N^{\omega} = \frac{1}{N}\sum_{i=1}^{N} \delta_{\tilde{\theta}^i(\omega_{I_1},\ldots,\omega_{I_N})^{\mathrm{per}}} \tag{3.14}$$

denote the *empirical process of N-tuples of words* in ω cut out by the successive excursions. Then (3.13) gives (recall the definition of P^* and P_g^* in Section 2.1)

$$\begin{aligned} F_N^{\beta,h,\omega}(g) &= \mathcal{N}(g)^N E_g^*\left(\exp\left[N \int_{\widetilde{E}} (\tilde{\pi}_1 R_N^{\omega})(dy) \log \phi_{\beta,h}(y)\right]\right) \\ &= \mathcal{N}(g)^N E_g^*\big(\exp\big[N\Phi_{\beta,h}\big(R_N^{\omega}\big)\big]\big) \\ &= E^*\big(\exp\big[N\big\{\Phi_{\beta,h}\big(R_N^{\omega}\big) - g m_{R_N^{\omega}}\big\}\big]\big) \end{aligned} \tag{3.15}$$

with $\Phi_{\beta,h}$ and $\phi_{\beta,h}$ defined in (3.6)–(3.7). Next, let

$$\overline{S}^{\mathrm{que}}(\beta, h; g) = \limsup_{N\to\infty} \frac{1}{N} \log F_N^{\beta,h,\omega}(g), \qquad g \in [0,\infty) \tag{3.16}$$

and note that the limsup exists and is constant (possibly infinity) ω-a.s. because it is measurable w.r.t. the tail sigma-algebra of ω (which is trivial). By (1.15), the left-hand side of (3.12) is a power series that converges for $g > g^{\mathrm{que}}(\beta, h)$ and diverges for $g < g^{\mathrm{que}}(\beta, h)$. Hence, we have

$$\sup\{g \in \mathbb{R} : \overline{S}^{\mathrm{que}}(\beta, h; g) > 0\} \le g^{\mathrm{que}}(\beta, h) \le \inf\{g \in \mathbb{R} : \overline{S}^{\mathrm{que}}(\beta, h; g) < 0\}. \tag{3.17}$$

In Section 3.3, we will see that $g \mapsto \overline{S}^{\mathrm{que}}(\beta, h; g)$ is continuous and strictly decreasing when finite, so that $\overline{S}^{\mathrm{que}}(\beta, h; g)$ changes sign precisely at $g = g^{\mathrm{que}}(\beta, h)$.

3. A *naive* application of Varadhan's lemma to (3.15)–(3.16) based on the quenched LDP in Theorem 2.3 yields that

$$\overline{S}^{\mathrm{que}}(\beta, h; g) = \log \mathcal{N}(g) + \sup_{Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})} [\Phi_{\beta,h}(Q) - I_g^{\mathrm{que}}(Q)]. \tag{3.18}$$

This variational formula brings us close to where we want, because Lemma 2.1 and the formulas for $I_g^{\mathrm{que}}(Q)$ given in Theorem 2.3 tell us that

$$\text{r.h.s. (3.18)} = \begin{cases} \sup_{Q \in \mathcal{R}} [\Phi_{\beta,h}(Q) - g m_Q - I^{\mathrm{ann}}(Q)], & \text{if } g \in (0, \infty), \\ \sup_{Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})} [\Phi_{\beta,h}(Q) - I^{\mathrm{que}}(Q)], & \text{if } g = 0, \end{cases} \tag{3.19}$$

which is the same as the variational formulas in (3.5) and (3.8), except that the suprema in (3.19) are not restricted to $\mathcal{C}^{\mathrm{fin}}$. Unfortunately, the application of Varadhan's lemma is *problematic*, because $Q \mapsto m_Q$ and $Q \mapsto \Phi_{\beta,h}(Q)$ are neither bounded nor continuous in the weak topology. The proof of (3.18)–(3.19) therefore requires an *approximation argument*, which is written out in Appendix B and is valid for $g \in (0, \infty)$. This approximation argument also shows how the restriction to $\mathcal{C}^{\mathrm{fin}}$ comes in. This restriction is needed to make the variational formulas proper, namely, it is shown in Appendix A that if I^{ann} is finite, then also $\Phi_{\beta,h}$ is finite. Thus, we have

$$\overline{S}^{\mathrm{que}}(\beta, h; g) = S^{\mathrm{que}}(\beta, h; g), \qquad g \in (0, \infty). \tag{3.20}$$

4. To include $g \in (-\infty, 0)$ in (3.20) we argue as follows. We see from (3.6)–(3.7) and (3.15) that $F_N^{\beta,h,\omega}(g) = \infty$ for $g \in (-\infty, 0)$, and so it follows from (3.16) that $\overline{S}^{\mathrm{que}}(\beta, h; g) = \infty$ for $g \in (-\infty, 0)$. Moreover, we have

$$S^{\mathrm{que}}(\beta, h; g) \ge \log\left(\frac{1}{2}\right) + \sup_{\rho' \in \mathcal{P}(\mathbb{N})} [-g m_{\rho'} - h(\rho' | \rho)], \tag{3.21}$$

which is obtained from (3.5)–(3.7) by picking $Q = q'^{\otimes \mathbb{N}}$ with $q'(dx_1, \dots, dx_m) = \rho'(m)\nu(dx_1) \times \cdots \times \nu(dx_m)$, $m \in \mathbb{N}$, $x_1, \dots, x_m \in \mathbb{R}$ [compare with (2.3)]. By picking $\rho'(m) = \delta_{m,L}$, $m \in \mathbb{N}$, with $L \in \mathbb{N}$ arbitrary, we get from (3.21) that $S^{\mathrm{que}}(\beta, h; g) \ge \log(\frac{1}{2}) - gL + \log \rho(L)$. Letting $L \to \infty$ and using (1.6), we obtain that $S^{\mathrm{que}}(\beta, h; g) = \infty$ for $g \in (-\infty, 0)$. Thus, (3.20) extends to

$$\overline{S}^{\mathrm{que}}(\beta, h; g) = S^{\mathrm{que}}(\beta, h; g), \qquad g \in \mathbb{R} \setminus \{0\}. \tag{3.22}$$

5. We next complete the proof of (i) and (ii). In Appendix C, we will show that

$$\overline{S}^{\mathrm{que}}(\beta, h; 0+) \geq S_*^{\mathrm{que}}(\beta, h), \tag{3.23}$$

where $\overline{S}^{\mathrm{que}}(\beta, h; 0+) = \lim_{g\downarrow 0} \overline{S}^{\mathrm{que}}(\beta, h; g)$. Moreover, by (3.5) and (3.20), we have

$$\overline{S}^{\mathrm{que}}(\beta, h; 0+) = S^{\mathrm{que}}(\beta, h; 0+) \leq S^{\mathrm{que}}(\beta, h; 0). \tag{3.24}$$

Furthermore, from (3.5) and (3.8) it follows that

$$\begin{aligned} S_*^{\mathrm{que}}(\beta, h) &= \sup_{Q\in\mathcal{C}^{\mathrm{fin}}} \left[\Phi_{\beta,h}(Q) - I^{\mathrm{que}}(Q)\right] \\ &\geq \sup_{Q\in\mathcal{C}^{\mathrm{fin}}\cap\mathcal{R}} \left[\Phi_{\beta,h}(Q) - I^{\mathrm{que}}(Q)\right] \\ &= S^{\mathrm{que}}(\beta, h; 0), \end{aligned} \tag{3.25}$$

where the last equality uses that $I^{\mathrm{que}} = I^{\mathrm{ann}}$ on $\mathcal{C}^{\mathrm{fin}} \cap \mathcal{R}$ [recall (2.18)]. Combining (3.23)–(3.25), we obtain

$$\overline{S}^{\mathrm{que}}(\beta, h; 0+) = S^{\mathrm{que}}(\beta, h; 0) = S_*^{\mathrm{que}}(\beta, h). \tag{3.26}$$

Combining (3.8), (3.17), and (3.26), we get (i) and (ii).

6. In Appendix A, we will prove that, for every $g \in (0,\infty)$, ω-a.s. there exists a $K(\omega, g) < \infty$ such that

$$\begin{aligned} -g m_{R_N^\omega} + \Phi_{\beta,h}\big(R_N^\omega\big) &\leq K(\omega, g) \\ &\forall N \in \mathbb{N}, 0 = k_0 < k_1 < \cdots < k_N < \infty. \end{aligned} \tag{3.27}$$

Via (3.15)–(3.16) this implies that $\overline{S}^{\mathrm{que}}(\beta, h; g) < \infty$ for $g \in (0,\infty)$.

7. By (3.5), $g \mapsto S^{\mathrm{que}}(\beta, h; g)$ is a supremum of functions that are finite and linear on $\mathbb{R}$. Hence, $g \mapsto S^{\mathrm{que}}(\beta, h; g)$ is lower semicontinuous and convex on $\mathbb{R}$ and, being finite on $(0,\infty)$, is continuous on $(0,\infty)$. Moreover, since $m_Q \geq 1$, it is strictly decreasing on $(0,\infty)$ as well (with right-derivative ≤ -1). This completes the proof of part (iii). $\square$

Figure 6 provides a sketch of $g \mapsto S^{\mathrm{que}}(\beta, h; g)$ for (β, h) drawn from $\mathcal{L}^{\mathrm{que}}$, $\partial\mathcal{D}^{\mathrm{que}}$ and $\mathrm{int}(\mathcal{D}^{\mathrm{que}})$, respectively, and completes the variational characterization in Theorem 3.1. In Section 3.3, we look at $h \mapsto S^{\mathrm{que}}(\beta, h; 0)$ and obtain the picture drawn in Figure 7, which is crucial for our analysis.

REMARK. In Section 6, we will show that

$$S^{\mathrm{que}}\left(\beta, h_c^{\mathrm{ann}}\left(\frac{\beta}{\alpha}\right); 0\right) \in (0,\infty]. \tag{3.28}$$

It will turn out that $S^{\mathrm{que}}(\beta, h_c^{\mathrm{ann}}(\frac{\beta}{\alpha}); 0) < \infty$ for some choices of ρ, but we do not know whether it is finite in general.

A major advantage of the variational formula in (3.8) over the one in (3.5) at $g = 0$ is that the supremum runs over $\mathcal{C}^{\mathrm{fin}}$ rather than $\mathcal{C}^{\mathrm{fin}} \cap \mathcal{R}$. This will be *crucial* for the proof of Corollaries 1.3–1.6 in Sections 5–7.

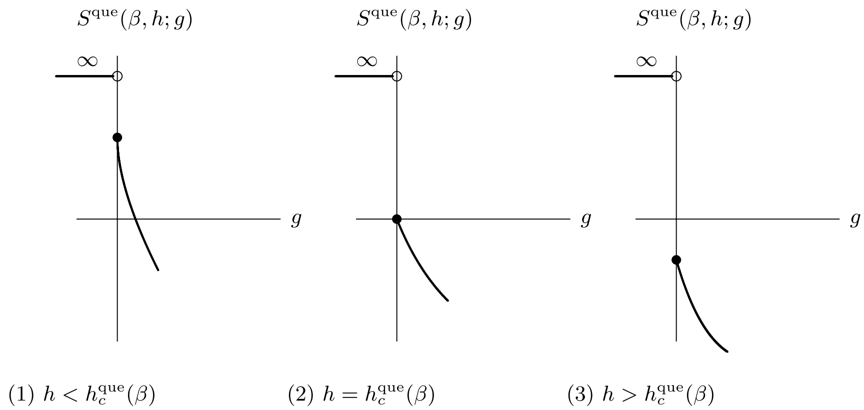

FIG. 6. *Qualitative picture of $g \mapsto S^{\mathrm{que}}(\beta, h; g)$ for $\beta, h > 0$. Picture* (1) *actually splits into two subcases*: *in Section* 3.3, *we will see that $S^{\mathrm{que}}(\beta, h; 0)$ is infinite when $h \in (0, h_c^{\mathrm{ann}}(\beta/\alpha))$ and finite when $h \in (h_c^{\mathrm{ann}}(\beta/\alpha), h_c^{\mathrm{que}}(\beta))$. At $h = h_c^{\mathrm{ann}}(\beta/\alpha)$, it can be either finite or infinity* (*see the remark at the end of Section* 3.1). *In Section* 3.3, *we will also see that $h \mapsto S^{\mathrm{que}}(\beta, h; 0)$ is continuous and strictly decreasing when finite.*

3.2. *Annealed excess free energy and critical curve.* In order to exploit Theorem 3.1, we need an analogous variational expression for the annealed excess free energy defined in (1.17)–(1.18). This variational expression will serve as a *comparison object* and will be crucial for the proof of Corollaries 1.2–1.4.

THEOREM 3.2. *Assume* (1.2) *and* (1.6)–(1.7). *Fix $\beta, h > 0$.*

(i) *The annealed excess free energy is given by*

$$g^{\mathrm{ann}}(\beta, h) = \inf\{g \in \mathbb{R} : S^{\mathrm{ann}}(\beta, h; g) < 0\}, \tag{3.29}$$

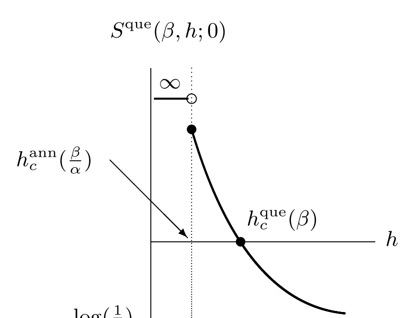

FIG. 7. *Qualitative picture of $h \mapsto S^{\mathrm{que}}(\beta, h; 0)$ for $\beta > 0$. In Section* 3.3, *we will see that $h \mapsto S^{\mathrm{que}}(\beta, h; 0)$ is strictly decreasing when finite and tends to $\log(\frac{1}{2})$ as $h \to \infty$. At $h = h_c^{\mathrm{ann}}(\frac{\beta}{\alpha})$ the value can be finite or infinite. We expect that $h \mapsto \overline{S}^{\mathrm{que}}(\beta, h; 0)$ coincides with $h \mapsto S^{\mathrm{que}}(\beta, h; 0)$*, (*see Section* 1.5, *item* 8).

where

$$S^{\text{ann}}(\beta,h;g)=\sup_{Q\in\mathcal{C}^{\text{fin}}}\big[\Phi_{\beta,h}(Q)-gm_Q-I^{\text{ann}}(Q)\big]. \tag{3.30}$$

(ii) *The function* $g\mapsto S^{\text{ann}}(\beta,h;g)$ *is lower semicontinuous, convex and non-increasing on* $\mathbb{R}$, *infinite on* $(-\infty,g^{\text{ann}}(\beta,h))$, *and finite, continuous and strictly decreasing on* $[g^{\text{ann}}(\beta,h),\infty)$.

PROOF. Throughout the proof $\beta,h>0$ are fixed.

(i) Replacing $\widetilde{Z}_n^{\beta,h,\omega}$ by $\mathbb{E}(\widetilde{Z}_n^{\beta,h,\omega})$ in (3.12)–(3.13) we define, in analogy with (3.16),

$$\overline{S}^{\text{ann}}(\beta,h;g)=\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}\big(F_N^{\beta,h,\omega}(g)\big). \tag{3.31}$$

Using (2.3)–(2.4), (3.7), (3.10) and (3.13) we compute

$$\overline{S}^{\text{ann}}(\beta,h;g)=\log\mathcal{N}(\beta,h;g) \tag{3.32}$$

with

$$\begin{aligned}\mathcal{N}(\beta,h;g)&=\int_{\widetilde{E}}q_{\rho,\nu}(dy)e^{-g\tau(y)}\phi_{\beta,h}(y)\\&=\sum_{m\in\mathbb{N}}\int_{x_1,\dots,x_m\in\mathbb{R}}\rho(m)\nu(dx_1)\times\cdots\times\nu(dx_m)\\&\qquad\qquad\times e^{-gm}\frac{1}{2}\big(1+e^{-2\beta hm-2\beta[x_1+\cdots+x_m]}\big)\\&=\frac{1}{2}\sum_{m\in\mathbb{N}}\rho(m)e^{-gm}+\frac{1}{2}\sum_{m\in\mathbb{N}}\rho(m)e^{-gm}\big[e^{-2\beta h+M(2\beta)}\big]^m\\&=\frac{1}{2}\mathcal{N}(g)+\frac{1}{2}\mathcal{N}\big(g-[M(2\beta)-2\beta h]\big),\end{aligned} \tag{3.33}$$

where $\mathcal{N}(g)$ is the normalization constant in (2.4). The right-hand side of (3.32) has the behavior as sketched in Figure 8. It is therefore immediate that (3.29)–(3.30) is consistent with (1.21), provided we have

$$S^{\text{ann}}(\beta,h;g)=\overline{S}^{\text{ann}}(\beta,h;g). \tag{3.34}$$

To prove this equality, we must distinguish three cases.

(I) $g\geq g^{\text{ann}}(\beta,h)=0\vee[M(2\beta)-2\beta h]$. The proof comes in 2 steps. Note that the right-hand side of (3.33) is finite.

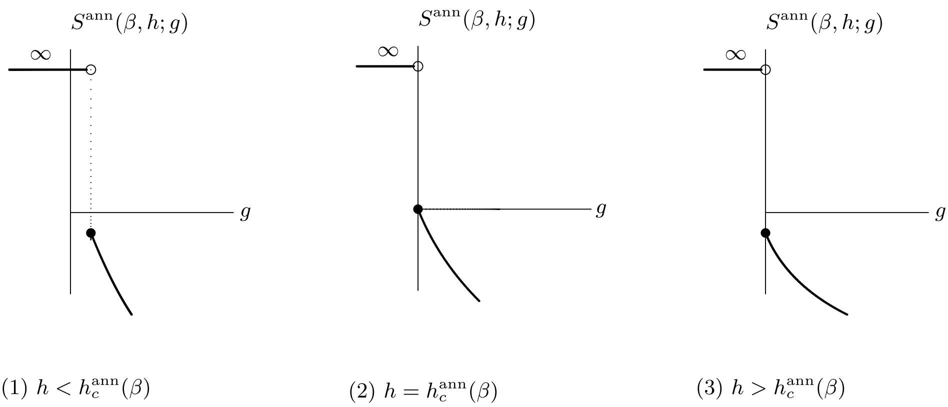

FIG. 8. *Qualitative picture of* $g \mapsto S^{\mathrm{ann}}(\beta,h;g)$ *for* $\beta, h > 0$. *Compare with Figure* 6. *In picture* (1), *the jump to infinity occurs at* $g = M(2\beta) - 2\beta h$.

1. Note that $\Phi_{\beta,h}(Q)$ defined in (3.6) is a functional of $\tilde{\pi}_1 Q$. Moreover, by (2.5),

$$\inf_{\substack{Q\in\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})\\ \tilde{\pi}_1 Q = q}} H\big(Q|q_{\rho,\nu}^{\otimes\mathbb{N}}\big) = h(q|q_{\rho,\nu}) \qquad \forall q \in \mathcal{P}(\widetilde{E}) \tag{3.35}$$

with the infimum *uniquely* attained at $Q = q^{\otimes\mathbb{N}}$, where the right-hand side denotes the relative entropy of q w.r.t. $q_{\rho,\nu}$. (The uniqueness of the minimum is easily deduced from the strict convexity of relative entropy on finite cylinders.) Consequently, the variational formula in (3.30) reduces to [recall (3.3)]

$$\begin{aligned} &S^{\mathrm{ann}}(\beta,h;g)\\ &\quad= \sup_{\substack{q\in\mathcal{P}(\widetilde{E})\\ m_q<\infty,\, h(q|q_{\rho,\nu})<\infty}} \left\{\int_{\widetilde{E}} q(dy)[-g\tau(y) + \log\phi_{\beta,h}(y)] - h(q|q_{\rho,\nu})\right\}\\ &\quad= \sup_{\substack{q\in\mathcal{P}(\widetilde{E})\\ m_q<\infty,\, h(q|q_{\rho,\nu})<\infty}} \left\{\int_{\widetilde{E}} q(dy)[-g\tau(y) + \log\phi_{\beta,h}(y)]\right.\\ &\qquad\qquad\qquad\left. - \int_{\widetilde{E}} q(dy)\log\left(\frac{q(dy)}{q_{\rho,\nu}(dy)}\right)\right\} \end{aligned} \tag{3.36}$$

with $\phi_{\beta,h}(y)$ defined in (3.7) and $m_q = \int_{\widetilde{E}} q(dy)\tau(y)$.

2. Define

$$q_{\beta,h;g}(dy) = \frac{1}{\mathcal{N}(\beta,h;g)} e^{-g\tau(y)}\phi_{\beta,h}(y) q_{\rho,\nu}(dy), \qquad y \in \widetilde{E} \tag{3.37}$$

with $\mathcal{N}(\beta, h; g)$ the normalizing constant in (3.33) (which is finite because $g \geq 0 \vee [M(2\beta) - 2\beta h] \geq [M(2\beta) - 2\beta h]$). Then the term between braces in the second equality of (3.36) can be rewritten as

$$\log \mathcal{N}(\beta, h; g) - h(q|q_{\beta,h;g}) \tag{3.38}$$

and so we have two cases:

(1) if both $m_{q_{\beta,h;g}} < \infty$ and $h(q_{\beta,h;g}|q_{\rho,\nu}) < \infty$, then the supremum in (3.36) has a unique maximizer at $q = q_{\beta,h;g}$;

(2) if $m_{q_{\beta,h;g}} = \infty$ and/or $h(q_{\beta,h;g}|q_{\rho,\nu}) = \infty$, then there is a maximizing sequence $(q_l)_{l\in\mathbb{N}}$ with $m_{q_l} < \infty$ and $h(q_l|q_{\rho,\nu}) < \infty$ for all $l \in \mathbb{N}$, that is, $\lim_{l\to\infty} h(q_l|q_{\beta,h;g}) = 0$ (and hence $\text{w-}\lim_{l\to\infty} q_l = q_{\beta,h;g}$ with w-lim denoting the weak limit).

In both cases,

$$S^{\text{ann}}(\beta, h; g) = \log \mathcal{N}(\beta, h; g), \tag{3.39}$$

which settles (3.34) in view of (3.32).

(II) $g < [M(2\beta) - 2\beta h]$. It follows from (3.32)–(3.33) that $\overline{S}^{\text{ann}}(\beta, h, g) = \infty$. We therefore need to show that $S^{\text{ann}}(\beta, h; g) = \infty$ as well. For $L \in \mathbb{N}$, let $q_\beta^L \in \mathcal{P}(\widetilde{E})$ be defined by

$$q_\beta^L(dx_1, \dots, dx_m) = \delta_{mL}\nu_\beta(dx_1) \times \cdots \times \nu_\beta(dx_m), \qquad m \in \mathbb{N}, x_1, \dots, x_m \in \mathbb{R}, \tag{3.40}$$

where $\nu_\beta \in \mathcal{P}(\mathbb{R})$ is defined by

$$\nu_\beta(dx) = e^{-2\beta x - M(2\beta)}\nu(dx), \qquad x \in \mathbb{R}. \tag{3.41}$$

Put $Q_\beta^L = (q_\beta^L)^{\otimes\mathbb{N}}$. Then $m_{Q_\beta^L} = L$, while

$$\begin{aligned} I^{\text{ann}}(Q_\beta^L) &= H(Q_\beta^L | q_{\rho,\nu}^{\otimes\mathbb{N}}) \\ &= h(q_\beta^L | q_{\rho,\nu}) \\ &= \int_{\widetilde{E}} q_\beta^L(dy) \log\left[\frac{dq_\beta^L}{dq_{\rho,\nu}}(y)\right] \\ &= -\log\rho(L) + Lh(\nu_\beta|\nu) \\ &= -\log\rho(L) + L\int_{\mathbb{R}} \nu_\beta(dx)\log(e^{-2\beta x - M(2\beta)}) \\ &= -\log\rho(L) - L[2\beta\mathbb{E}_{\nu_\beta}(\omega_1) + M(2\beta)] \end{aligned} \tag{3.42}$$

and

$$\begin{aligned}\Phi_{\beta,h}(Q^L_\beta) &= \int_{\widetilde{E}} q^L_\beta(dy)\log\phi_{\beta,h}(y)\\ &= \int_{\mathbb{R}^L} \nu_\beta(dx_1)\times\cdots\times\nu_\beta(dx_L)\\ &\qquad\times\log\left(\frac{1}{2}\left[1+e^{-2\beta hL-2\beta[x_1+\cdots+x_L]}\right]\right)\\ &\geq \log\left(\frac{1}{2}\right) - L\left[2\beta\mathbb{E}_{\nu_\beta}(\omega_1)+2\beta h\right].\end{aligned} \tag{3.43}$$

It follows that

$$\begin{aligned}&\Phi_{\beta,h}(Q^L_\beta) - gm_{Q^L_\beta} - I^{\mathrm{ann}}(Q^L_\beta)\\ &\qquad\geq \log(\tfrac{1}{2}) + \log\rho(L) + L\left[M(2\beta)-2\beta h - g\right],\end{aligned} \tag{3.44}$$

which tends to infinity as $L\to\infty$ [use (1.6) and let $L\to\infty$ along the support of ρ].

(III) $M(2\beta)-2\beta h<0$ and $g\in[M(2\beta)-2\beta h, 0)$. Repeat the argument in (3.42)–(3.44) with Q^L_β replaced by Q^L_0 and keep only the first term in the right-hand side of (3.44). This gives

$$\Phi_{\beta,h}(Q^L_0) - gm_{Q^L_0} - I^{\mathrm{ann}}(Q^L_0) \geq \log(\tfrac{1}{2}) + \log\rho(L) - Lg, \tag{3.45}$$

which tends to infinity as $L\to\infty$ for $g<0$. $\square$

Figure 8 provides a sketch of $g\mapsto S^{\mathrm{ann}}(\beta,h;g)$ for (β,h) drawn from $\mathcal{L}^{\mathrm{ann}}$, $\partial\mathcal{D}^{\mathrm{ann}}$ and $\mathrm{int}(\mathcal{D}^{\mathrm{ann}})$, respectively, and completes the variational characterization in Theorem 3.2. Figure 9 provides a sketch of $h\mapsto S^{\mathrm{ann}}(\beta,h;0)$.

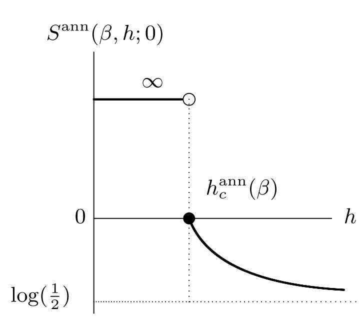

FIG. 9. *Qualitative picture of $h\mapsto S^{\mathrm{ann}}(\beta,h;0)$ for $\beta>0$. Compare with Figure 7. It follows from (3.33) that $\lim_{h\to\infty} S^{\mathrm{ann}}(\beta,h;0)=\log(\frac{1}{2})$. Since $S^{\mathrm{que}}\leq S^{\mathrm{ann}}$, the same is true for S^{que}, as claimed in Figure 7.*

3.3. *Proof of Theorem* 1.1. Theorems 3.1 and 3.2 complete the proof of part (i) of Theorem 1.1. From the computations carried out in Section 3.2, we also get parts (ii) and (iii) for the annealed model, but to get parts (ii) and (iii) for the quenched model we need some further information.

Recall from (3.4) that

$$g^{\mathrm{que}}(\beta,h)=\inf\{g\in\mathbb{R}: S^{\mathrm{que}}(\beta,h;g)<0\}. \tag{3.46}$$

It therefore follows from (1.12) that

$$\begin{aligned} h_c^{\mathrm{que}}(\beta) &= \inf\{h\ge 0: g^{\mathrm{que}}(\beta,h)=0\} \\ &= \inf\{h\ge 0: \inf\{g\in\mathbb{R}: S^{\mathrm{que}}(\beta,h;g)<0\}=0\} \\ &= \inf\Big\{h\ge 0: \lim_{g\downarrow 0} S^{\mathrm{que}}(\beta,h;g)\le 0\Big\} \\ &= \inf\{h\ge 0: S^{\mathrm{que}}(\beta,h;0)\le 0\}. \end{aligned} \tag{3.47}$$

The third equality uses that the map $g\mapsto S^{\mathrm{que}}(\beta,h;g)$ is decreasing on $[0,\infty)$. The fourth uses that $S^{\mathrm{que}}(\beta,h;0)=S^{\mathrm{que}}(\beta,h;0+)$. This implies that as far as the critical curve is concerned we do not need $\overline{S}^{\mathrm{que}}(\beta,h;0)$, but rather $\overline{S}^{\mathrm{que}}(\beta,h;0+)=S^{\mathrm{que}}(\beta,h;0)$.

Theorem 3.1 provides no information on $S^{\mathrm{que}}(\beta,h;0)$. We know that, for every $\beta>0$, $h\mapsto S^{\mathrm{que}}(\beta,h;0)$ is lower semicontinuous, convex and nonincreasing on $(0,\infty)$. Indeed, $h\mapsto \log\phi_{\beta,h}(y)$ is continuous, convex and nonincreasing for all $y\in\widetilde{E}$, hence $h\mapsto \Phi_{\beta,h}(Q)$ is lower semicontinuous, convex and nonincreasing for every $Q\in\mathcal{C}^{\mathrm{fin}}$, and these properties are preserved under taking suprema. We know that $h\mapsto S^{\mathrm{que}}(\beta,h;0)$ is strictly negative on $(h_c^{\mathrm{que}}(\beta),\infty)$ [because it is convex, is zero at $h_c^{\mathrm{que}}(\beta)$ and tends to $\log(\frac{1}{2})$ as $h\to\infty$]. In Section 6, we prove the following theorem, which corroborates the picture drawn in Figure 7 and completes the proof of parts (ii) and (iii) of Theorem 1.1 for the quenched model.

THEOREM 3.3. *For every* $\beta>0$,

$$S^{\mathrm{que}}(\beta,h;0)=S_*^{\mathrm{que}}(\beta,h)\begin{cases} =\infty, & \textit{for } h<h_c^{\mathrm{ann}}(\beta/\alpha),\\ \in(0,\infty], & \textit{for } h=h_c^{\mathrm{ann}}(\beta/\alpha),\\ \in\big(\log(\frac{1}{2}),\infty\big), & \textit{for } h>h_c^{\mathrm{ann}}(\beta/\alpha).\end{cases} \tag{3.48}$$

We close this section with the following remark. The difference between the variational formulas in (3.5) (quenched model) and (3.30) (annealed model) is that the supremum in the former runs over $\mathcal{C}^{\mathrm{fin}}\cap\mathcal{R}$ while the supremum in the latter runs over $\mathcal{C}^{\mathrm{fin}}$. Both involve the annealed rate function I^{ann}. However, the restriction to $\mathcal{R}$ for the quenched model allows us to replace I^{ann} by I^{que} [recall (2.18)]. After passing to the limit $g\downarrow 0$, we can remove the restriction to $\mathcal{R}$ to obtain the alternative variational formula for the quenched model given in (3.8). The latter turns out to be crucial in Sections 5 and 6.

Note that the two variational formulas for $g \neq 0$ are different even when $\alpha = 1$, although in that case $I^{\text{ann}} = I^{\text{que}}$ (compare Theorems 2.2 and 2.3). *For $\alpha = 1$ the quenched and the annealed critical curves coincide, but the free energies do not*. We will see in Sections 4–8 that the continuity of $g \mapsto S^{\text{que}}(\beta, h; g)$ at $g = 0$ and the equality $S^{\text{que}}(\beta, h; 0) = S_*^{\text{que}}(\beta, h)$, given by the variational formula in (3.8), are *essential ingredients of the variational approach* to the copolymer model.

4. Proof of Corollary 1.2.

PROOF. The claim is trivial for $h_c^{\text{que}}(\beta) \le h < h_c^{\text{ann}}(\beta)$ because $g^{\text{que}}(\beta, h) = 0 < g^{\text{ann}}(\beta, h)$. Therefore, we may assume that $0 < h < h_c^{\text{que}}(\beta)$. Since $I^{\text{que}}(Q) \ge I^{\text{ann}}(Q)$, (3.5) and (3.30) yield

$$S^{\text{que}}(\beta, h; g) \le S^{\text{ann}}(\beta, h; g), \tag{4.1}$$

which via (3.4) and (3.29), implies that $g^{\text{que}}(\beta, h) \le g^{\text{ann}}(\beta, h)$, a property that is also evident from (1.15) and (1.18). To prove that $g^{\text{que}}(\beta, h) < g^{\text{ann}}(\beta, h)$ for $0 < h < h_c^{\text{que}}(\beta)$, we combine (4.1) with Figures 6 and 8. First note that

$$\begin{aligned} S^{\text{que}}\big(\beta, h; g^{\text{ann}}(\beta, h)\big) \le S^{\text{ann}}\big(\beta, h; g^{\text{ann}}(\beta, h)\big) < 0,& \\ 0 < h < h_c^{\text{ann}}(\beta).& \end{aligned} \tag{4.2}$$

Next, for $0 < h < h_c^{\text{ann}}(\beta)$, $g \mapsto S^{\text{ann}}(\beta, h; g)$ blows up at $g = g^{\text{ann}}(\beta, h) > 0$ by jumping from a strictly negative value to infinity (see Figure 8). Since $S^{\text{que}}(\beta, h; g^{\text{ann}}(\beta, h)) < 0$, and $g \mapsto S^{\text{que}}(\beta, h; g)$ is strictly decreasing and continuous when finite, the claim is immediate from Theorem 1.1(ii), which says that $S^{\text{que}}(\beta, h; g^{\text{que}}(\beta, h)) = 0$. $\square$

5. Proof of Corollary 1.3.

PROOF. Throughout the proof, $\alpha > 1$ and $\beta > 0$ are fixed. It follows from (2.5) and the remark made below it that

$$H\big(Q|q_{\rho,\nu}^{\otimes\mathbb{N}}\big) \ge h(\tilde{\pi}_1 Q|q_{\rho,\nu}), \qquad H\big(\Psi_Q|\nu^{\otimes\mathbb{N}}\big) \ge h(\pi_1\Psi_Q|\nu), \tag{5.1}$$

where $\tilde{\pi}_1$ is the projection onto the first *word* and π_1 is the projection onto the first *letter*. Moreover, it follows from (2.11) that

$$\pi_1\Psi_Q = \pi_1\Psi_{(\tilde{\pi}_1 Q)^{\otimes\mathbb{N}}}. \tag{5.2}$$

Since $m_Q = m_{(\tilde{\pi}_1 Q)^{\otimes\mathbb{N}}} = m_{(\tilde{\pi}_1 Q)}$, (5.1)–(5.2) combine with (3.8) to give

$$\begin{aligned} S_*^{\text{que}}(\beta, h) \le \sup_{\substack{q\in\mathcal{P}(\widetilde{E})\\ m_q<\infty, h(q|q_{\rho,\nu})<\infty}} &\bigg[\int_{\widetilde{E}} q(dy)\log\phi_{\beta,h}(y) \\ &\quad - h(q|q_{\rho,\nu}) - (\alpha-1)m_q h(\pi_1\psi_q|\nu)\bigg], \end{aligned} \tag{5.3}$$

where

$$\begin{aligned}\phi_{\beta,h}(y) &= \tfrac{1}{2}\big(1+e^{-2\beta h m-2\beta[x_1+\cdots+x_m]}\big),\\ q_{\rho,\nu}(dy) &= \rho(m)\nu(dx_1)\times\cdots\times\nu(dx_m)\end{aligned} \tag{5.4}$$

and

$$(\pi_1\psi_q)(dx) = \frac{1}{m_q}\sum_{m\in\mathbb{N}}\sum_{k=1}^{m} q_m\big(E^{k-1},dx,E^{m-k}\big) \tag{5.5}$$

with the notation

$$q(dy) = q_m(dx_1,\ldots,dx_m), \qquad y=(x_1,\ldots,x_m). \tag{5.6}$$

Let

$$q^*_{\beta,h}(dy) = \frac{1}{\mathcal{N}(\beta,h)}\phi_{\beta,h}(y)q_{\rho,\nu}(dy) \tag{5.7}$$

with $\mathcal{N}(\beta,h)$ the normalizing constant [which equals $\mathcal{N}(\beta,h;0)$ in (3.33) and is finite for $h\ge h_c^{\mathrm{ann}}(\beta)=M(2\beta)/2\beta$]. Therefore, combining the first two terms in the supremum in (5.3), we obtain

$$S_*^{\mathrm{que}}(\beta,h)\le \log\mathcal{N}(\beta,h) - \inf_{\substack{q\in\mathcal{P}(\widetilde{E})\\ m_q<\infty}}\big[h\big(q|q^*_{\beta,h}\big)+(\alpha-1)m_q h(\pi_1\psi_q|\nu)\big], \tag{5.8}$$

where we drop the entropy constraint because it is no longer neeeded. Since $\mathcal{N}(\beta,h_c^{\mathrm{ann}}(\beta))=1$, we have

$$S_*^{\mathrm{que}}\big(\beta,h_c^{\mathrm{ann}}(\beta)\big)\le -\inf_{\substack{q\in\mathcal{P}(\widetilde{E})\\ m_q<\infty}}\big[h\big(q|q^*_{\beta,h_c^{\mathrm{ann}}(\beta)}\big)+(\alpha-1)m_q h(\pi_1\psi_q|\nu)\big]. \tag{5.9}$$

The first term achieves its minimal value zero uniquely at $q=q^*_{\beta,h_c^{\mathrm{ann}}(\beta)}$ (or along a minimizing sequence converging to $q^*_{\beta,h_c^{\mathrm{ann}}(\beta)}$). However, $\pi_1\psi_{q^*_{\beta,h_c^{\mathrm{ann}}(\beta)}}=\frac{1}{2}\nu+\frac{1}{2}\nu_\beta\neq\nu$ [recall (3.41)], and so the second term is not zero (or does not converge to zero), so that we have

$$S_*^{\mathrm{que}}\big(\beta,h_c^{\mathrm{ann}}(\beta)\big)<0. \tag{5.10}$$

Since $S_*^{\mathrm{que}}(\beta,h_c^{\mathrm{que}}(\beta))=0$ and $h\mapsto S_*^{\mathrm{que}}(\beta,h)$ is strictly decreasing on $(h_c^{\mathrm{ann}}(\beta/\alpha),\infty)$, it follows that $h_c^{\mathrm{que}}(\beta)<h_c^{\mathrm{ann}}(\beta)$. □

We close this section with the following remark. As (2.17) shows, $I^{\mathrm{fin}}(Q)$ depends on $q_{\rho,\nu}$, the reference law defined in (2.3). Since the latter depends on the full law $\rho\in\mathcal{P}(\mathbb{N})$ of the excursion lengths, it is evident from Theorem 1.1(iii) and (3.8) that the quenched critical curve is *not* a function of the exponent α in (1.6) alone. This supports the statement made in Section 1.5, item 6.

6. Proof of Corollary 1.4. The proof is immediate from Theorem 3.3 (recall Figure 7), which is proved in Sections 6.1–6.3.

6.1. *Proof for $h > h_c^{\mathrm{ann}}(\beta/\alpha)$.* In what follows, we take $g \in [0, \infty)$, so that $\mathcal{N}(g) < \infty$.

PROOF. Recall from (3.15)–(3.16) that

$$\begin{aligned}\overline{S}^{\mathrm{que}}(\beta, h; g) &= \limsup_{N\to\infty} \frac{1}{N} \log F_N^{\beta,h,\omega}(g) \\ &= \log \mathcal{N}(g) + \limsup_{N\to\infty} \frac{1}{N} \log E_g^*\big(\exp\big[N\Phi_{\beta,h}\big(R_N^\omega\big)\big]\big).\end{aligned} \tag{6.1}$$

Abbreviate

$$S_N^\omega(g) = E_g^*\big(\exp\big[N\Phi_{\beta,h}\big(R_N^\omega\big)\big]\big) \tag{6.2}$$

and pick

$$t = [0, 1], \qquad h = h_c^{\mathrm{ann}}(\beta t). \tag{6.3}$$

Then the tth moment of $S_N^\omega(g)$ can be estimated as [recall (3.10)–(3.11)]

$$\begin{aligned}\mathbb{E}\big(\big[S_N^\omega(g)\big]^t\big) &= \mathbb{E}\left(\left[E_g^*\left(\exp\left[\sum_{i=1}^N \log\left(\frac{1}{2}\big[1 + e^{-2\beta\sum_{k\in I_i}(\omega_k+h)}\big]\right)\right]\right)\right]^t\right) \\ &= \mathbb{E}\left(\left[E_g^*\left(\prod_{i=1}^N \frac{1}{2}\big[1 + e^{-2\beta\sum_{k\in I_i}(\omega_k+h)}\big]\right)\right]^t\right) \\ &= \mathbb{E}\left(\left[\sum_{0<k_1<\cdots<k_N<\infty} \left\{\prod_{i=1}^N \rho_g(k_i - k_{i-1})\right\}\right.\right. \\ &\qquad\qquad \left.\left.\times \left\{\prod_{i=1}^N \frac{1}{2}\big[1 + e^{-2\beta\sum_{k\in(k_{i-1},k_i]}(\omega_k+h)}\big]\right\}\right]^t\right) \\ &\le \mathbb{E}\left(\sum_{0<k_1<\cdots<k_N<\infty} \left\{\prod_{i=1}^N \rho_g(k_i - k_{i-1})^t\right\}\right. \\ &\qquad\qquad \left.\times \left\{\prod_{i=1}^N 2^{-t}\big[1 + e^{-2\beta t\sum_{k\in(k_{i-1},k_i]}(\omega_k+h)}\big]\right\}\right) \\ &= \sum_{0<k_1<\cdots<k_N<\infty} \left\{\prod_{i=1}^N \rho_g(k_i - k_{i-1})^t\right\} \\ &\qquad\qquad \times \left\{\prod_{i=1}^N 2^{-t}\big[1 + e^{(k_i-k_{i-1})[M(2\beta t)-2\beta t h]}\big]\right\}\end{aligned} \tag{6.4}$$

$$
\begin{aligned}
&= 2^{(1-t)N} \sum_{0<k_1<\cdots<k_N<\infty} \left\{ \prod_{i=1}^{N} \rho_g(k_i - k_{i-1})^t \right\} \\
&= \left(2^{1-t} \sum_{m\in\mathbb{N}} \rho_g(m)^t \right)^N .
\end{aligned}
$$

The inequality uses that $(u+v)^t \le u^t + v^t$ for $u, v \ge 0$ and $t \in [0,1]$, while the fifth equality uses that $M(2\beta t) - 2\beta t h = 0$ for the choice of t and h in (6.3) [recall (1.22)].

Let $K(g)$ denote the term between round brackets in the last line of (6.4). Then, for every $\epsilon > 0$, we have

$$
\begin{aligned}
&\mathbb{P}\left(\frac{1}{N} \log S_N^\omega(g) \ge \frac{1}{t} [\log K(g) + \epsilon] \right) \\
&\qquad = \mathbb{P}\big([S_N^\omega(g)]^t \ge K(g)^N e^{N\epsilon} \big) \\
&\qquad \le \mathbb{E}\big([S_N^\omega(g)]^t \big) K(g)^{-N} e^{-N\epsilon} \le e^{-N\epsilon}.
\end{aligned} \tag{6.5}
$$

Since this bound is summable, it follows from the Borel–Cantelli lemma that

$$
\limsup_{N\to\infty} \frac{1}{N} \log S_N^\omega(g) \le \frac{1}{t} \log K(g) \qquad \omega\text{-a.s.} \tag{6.6}
$$

Combine (6.1)–(6.2) and (6.6) to obtain

$$
\begin{aligned}
\overline{S}^{\mathrm{que}}(\beta, h; g) &\le \log \mathcal{N}(g) + \frac{1-t}{t} \log 2 + \frac{1}{t} \log\left(\sum_{m\in\mathbb{N}} \rho_g(m)^t \right) \\
&= \frac{1-t}{t} \log 2 + \frac{1}{t} \log\left(\sum_{m\in\mathbb{N}} e^{-gtm} \rho(m)^t \right).
\end{aligned} \tag{6.7}
$$

We see from (6.7) that $\overline{S}^{\mathrm{que}}(\beta, h_c^{\mathrm{ann}}(\beta t); g) < \infty$ for $g > 0$ and $t \in (0,1]$, and also for $g = 0$ and $t \in (1/\alpha, 1]$, that is, $\overline{S}^{\mathrm{que}}(\beta, h; 0) < \infty$ for $h \in (h_c^{\mathrm{ann}}(\beta/\alpha), h_c^{\mathrm{ann}}(\beta)]$. This completes the proof because we already know that $S^{\mathrm{que}}(\beta, h; 0) < 0$ for $h \in (h_c^{\mathrm{ann}}(\beta), \infty)$. $\square$

Note that if $\sum_{m\in\mathbb{N}} \rho(m)^{1/\alpha} < \infty$, then

$$
S^{\mathrm{que}}\big(\beta, h_c^{\mathrm{ann}}(\beta/\alpha); 0\big) \le \overline{S}^{\mathrm{que}}\big(\beta, h_c^{\mathrm{ann}}(\beta/\alpha); 0\big) < \infty.
$$

This explains the remark made below (3.28). The above argument also shows that $S^{\mathrm{que}}(\beta, h; g) = \overline{S}^{\mathrm{que}}(\beta, h; g) < \infty$ for all $\beta, h, g > 0$, since for $\beta, g > 0$ and any $h > h_c^{\mathrm{ann}}(\beta) = M(2\beta)/2\beta$ the fifth equality in (6.4) becomes an inequality for any $t \in (0,1]$, while any $0 < h \le h_c^{\mathrm{ann}}(\beta)$ equals $h = h_c^{\mathrm{ann}}(\beta t)$ for some $t \in (0,1]$.

6.2. *Proof for* $h < h_c^{\mathrm{ann}}(\beta/\alpha)$.

PROOF. For $L \in \mathbb{N}$, define [recall (3.41)]

$$q_\beta^L(dx_1, \dots, x_m) = \delta_{m,L}\,\nu_{\beta/\alpha}(dx_1) \times \cdots \times \nu_{\beta/\alpha}(dx_m), \qquad m \in \mathbb{N}, x_1, \dots, x_m \in \mathbb{R} \tag{6.8}$$

and

$$Q_\beta^L = (q_\beta^L)^{\otimes\mathbb{N}} \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}). \tag{6.9}$$

We will show that

$$h < h_c^{\mathrm{ann}}(\beta/\alpha) \quad \Longrightarrow \quad \liminf_{L\to\infty} \frac{1}{L}\left[\Phi_{\beta,h}(Q_\beta^L) - I^{\mathrm{que}}(Q_\beta^L)\right] > 0, \tag{6.10}$$

which will imply the claim because $Q_\beta^L \in \mathcal{C}^{\mathrm{fin}}$. [Recall (3.3) and note that both $m_{Q_\beta^L} = L$ and $I^{\mathrm{ann}}(Q_\beta^L) = h(q_\beta^L | q_{\rho,\nu}) = -\log\rho(L) + h(\nu_{\beta/\alpha}|\nu)L$ are finite.]

We have [recall (3.6) and (3.7)]

$$\begin{aligned} \Phi_{\beta,h}(Q_\beta^L) &= \int_{\widetilde{E}} q_\beta^L(dy) \log \phi_{\beta,h}(y), \\ H(Q_\beta^L | q_{\rho,\nu}^{\otimes\mathbb{N}}) = h(q_\beta^L | q_{\rho,\nu}) &= \int_{\widetilde{E}} q_\beta^L(dy) \log\left(\frac{q_\beta^L(dy)}{q_{\rho,\nu}(dy)}\right). \end{aligned} \tag{6.11}$$

Dropping the 1 in front of the exponential in (3.7), we obtain [similarly as in (3.42)–(3.44)]

$$\begin{aligned} &\Phi_{\beta,h}(Q_\beta^L) - H(Q_\beta^L | q_{\rho,\nu}^{\otimes\mathbb{N}}) \\ &\quad \geq \log\left(\frac{1}{2}\right) + \int_{\widetilde{E}} q_\beta^L(dy) \log\left[\frac{e^{-2\beta h\tau(y) - 2\beta\sigma(y)} q_{\rho,\nu}(dy)}{q_\beta^L(dy)}\right] \\ &\quad = \log\left(\frac{1}{2}\right) + \int_{\mathbb{R}^L} \nu_{\beta/\alpha}^{\otimes L}(dx_1, \dots, dx_L) \\ &\qquad\qquad \times \log\left[e^{-2\beta h L} e^{-2\beta[x_1 + \cdots + x_L]} \frac{d\nu^{\otimes L}}{d\nu_{\beta/\alpha}^{\otimes L}}(x_1, \dots, x_L)\rho(L)\right] \\ &\quad = \log\left(\frac{1}{2}\right) + \int_{\mathbb{R}^L} \nu_{\beta/\alpha}^{\otimes L}(dx_1, \dots, dx_L) \\ &\qquad\qquad \times \log\left[e^{[M(2\beta) - 2\beta h]L} \frac{d\nu_\beta^{\otimes L}}{d\nu_{\beta/\alpha}^{\otimes L}}(x_1, \dots, x_L)\rho(L)\right] \\ &\quad = \log\left(\frac{1}{2}\right) + [M(2\beta) - 2\beta h]L - h(\nu_{\beta/\alpha}|\nu_\beta)L + \log\rho(L). \end{aligned} \tag{6.12}$$

Furthermore, from (6.8) we have [recall (2.11)]

$$m_{Q_\beta^L} = L, \qquad \Psi_{Q_\beta^L} = \nu_{\beta/\alpha}^{\otimes\mathbb{N}}, \tag{6.13}$$

which gives

$$(\alpha-1) m_{Q_\beta^L} H\big(\Psi_{Q_\beta^L} | \nu^{\otimes\mathbb{N}}\big) = (\alpha-1) L h(\nu_{\beta/\alpha}|\nu). \tag{6.14}$$

Combining (6.12)–(6.14), recalling (2.16)–(2.17) and using that

$$\lim_{L\to\infty} L^{-1} \log \rho(L) = 0$$

by (1.6) when $L\to\infty$ along the support of ρ, we arrive at

$$\begin{aligned}
&\liminf_{L\to\infty} \frac{1}{L}\big[\Phi_{\beta,h}(Q_\beta^L) - I^{\mathrm{que}}(Q_\beta^L)\big] \\
&\quad \geq \big[M(2\beta) - 2\beta h\big] - h(\nu_{\beta/\alpha}|\nu_\beta) - (\alpha-1)h(\nu_{\beta/\alpha}|\nu) \\
&\quad = \alpha M\left(\frac{2\beta}{\alpha}\right) - 2\beta h = 2\beta\big[h_c^{\mathrm{ann}}(\beta/\alpha) - h\big],
\end{aligned} \tag{6.15}$$

where the first equality uses the relation [recall (1.22) and (3.41)]

$$\begin{aligned}
&h(\nu_{\beta/\alpha}|\nu_\beta) + (\alpha-1)h(\nu_{\beta/\alpha}|\nu) \\
&\quad = \int_{l\in\mathbb{R}} \nu_{\beta/\alpha}(dl)\bigg(\bigg[-\frac{2\beta}{\alpha}l - M\left(\frac{2\beta}{\alpha}\right)\bigg] + \big[2\beta l + M(2\beta)\big] \\
&\qquad\qquad + (\alpha-1)\bigg[-\frac{2\beta}{\alpha}l - M\left(\frac{2\beta}{\alpha}\right)\bigg]\bigg) \\
&\quad = M(2\beta) - \alpha M\left(\frac{2\beta}{\alpha}\right).
\end{aligned} \tag{6.16}$$

Note that (6.15) proves (6.10). □

6.3. *Proof for $h = h_c^{\mathrm{ann}}(\beta/\alpha)$.*

PROOF. Our starting point is (3.8), where (recall Theorem 2.3)

$$I^{\mathrm{que}}(Q) = I^{\mathrm{fin}}(Q) = H\big(Q|q_{\rho,\nu}^{\otimes\mathbb{N}}\big) + (\alpha-1) m_Q H\big(\Psi_Q|\nu^{\otimes\mathbb{N}}\big), \qquad Q \in \mathcal{C}^{\mathrm{fin}}. \tag{6.17}$$

The proof comes in 4 steps.

1. As shown in Birkner, Greven and den Hollander [4], equation (1.32),

$$H\big(Q|q_{\rho,\nu}^{\otimes\mathbb{N}}\big) = m_Q H\big(\Psi_Q|\nu^{\otimes\mathbb{N}}\big) + R(Q), \tag{6.18}$$

where $R(Q)\geq 0$ is the "specific relative entropy w.r.t. $\rho^{\otimes\mathbb{N}}$ of the word length process under Q conditional on the concatenation." Combining (6.17)–(6.18), we have $I^{\text{que}}(Q) \le \alpha H(Q|q_{\rho,\nu}^{\otimes\mathbb{N}})$, which yields

$$S_*^{\text{que}}(\beta,h) \ge \sup_{Q\in\mathcal{C}^{\text{fin}}} \big[\Phi_{\beta,h}(Q) - \alpha H\big(Q|q_{\rho,\nu}^{\otimes\mathbb{N}}\big)\big]. \tag{6.19}$$

2. The variational formula in the right-hand side of (6.19) can be computed similarly as in part (I) of Section 3.2. Indeed,

$$\text{r.h.s. (6.19)} = \sup_{\substack{q\in\mathcal{P}(\widetilde{E})\\ m_q<\infty, h(q|q_{\rho,\nu})<\infty}} \bigg[\int_{\widetilde{E}} q(dy)\log\phi_{\beta,h}(y) - \alpha h(q|q_{\rho,\nu})\bigg]. \tag{6.20}$$

Define

$$q_{\beta,h}(dy) = \frac{1}{\mathcal{N}(\beta,h)}\big[\phi_{\beta,h}(y)\big]^{1/\alpha} q_{\rho,\nu}(dy), \tag{6.21}$$

where $\mathcal{N}(\beta,h)$ is the normalizing constant. Then the term between square brackets in the right-hand side of (6.20) equals $\alpha\log\mathcal{N}(\beta,h) - \alpha h(q|q_{\beta,h})$, and hence [pick $q = q_{\beta,h}$ or a minimizing sequence; recall the two cases below (3.38)]

$$S_*^{\text{que}}(\beta,h) \ge \alpha\log\mathcal{N}(\beta,h), \tag{6.22}$$

provided $\mathcal{N}(\beta,h) < \infty$ so that $q_{\beta,h}$ is well defined.

3. Abbreviate $\mu = 2\beta/\alpha$. Since $h_c^{\text{ann}}(\beta/\alpha) = M(\mu)/\mu$, we have

$$\begin{aligned} &\mathcal{N}\big(\beta, h_c^{\text{ann}}(\beta/\alpha)\big)\\ &\quad = \sum_{m\in\mathbb{N}} \rho(m)\int_{\mathbb{R}^m} \nu(dx_1)\times\cdots\times\nu(x_m)\\ &\qquad\qquad \times \bigg\{\frac{1}{2}\big(1+e^{-\alpha(M(\mu)m+\mu[x_1+\cdots+x_m])}\big)\bigg\}^{1/\alpha}. \end{aligned} \tag{6.23}$$

Let Z be the random variable on $(0,\infty)$ whose law P is equal to the law of $e^{-(M(\mu)m+\mu[x_1+\cdots+x_m])}$ under $\rho(m)\nu(dx_1)\times\cdots\times\nu(x_m)$. Let

$$f_\alpha(z) = \big\{\tfrac{1}{2}(1+z^\alpha)\big\}^{1/\alpha}, \qquad z>0. \tag{6.24}$$

Then

$$\text{r.h.s. (6.23)} = E\big(f_\alpha(Z)\big). \tag{6.25}$$

We have $E(Z) = 1$. Moreover, an easy computation gives

$$\begin{aligned} f_\alpha'(z) &= \big(\tfrac{1}{2}\big)^{1/\alpha}\big(1+z^\alpha\big)^{(1/\alpha)-1} z^{\alpha-1},\\ f_\alpha''(z) &= \big(\tfrac{1}{2}\big)^{1/\alpha}\big(1+z^\alpha\big)^{(1/\alpha)-2} z^{\alpha-2}(\alpha-1), \end{aligned} \tag{6.26}$$

so that f_α is strictly convex. Therefore, by Jensen's inquality and the fact that P is not a point mass, we have

$$E\big(f_\alpha(Z)\big) > f_\alpha\big(E(Z)\big) = f_\alpha(1) = 1. \tag{6.27}$$

Combining (6.22)–(6.25) and (6.27), we arrive at

$$S_*^{\mathrm{que}}\big(\beta, h_c^{\mathrm{ann}}(\beta/\alpha)\big) > 0, \tag{6.28}$$

which proves the claim.

4. It remains to check that $\mathcal{N}(\beta, h_c^{\mathrm{ann}}(\beta/\alpha)) < \infty$. But $f_\alpha(z) \le (\frac{1}{2})^{1/\alpha}(1+z)$, $z > 0$, and so we have

$$\mathcal{N}\big(\beta, h_c^{\mathrm{ann}}(\beta/\alpha)\big) \le \big(\tfrac{1}{2}\big)^{1/\alpha}\big(1 + E(Z)\big) \le 2^{1-(1/\alpha)} < \infty. \tag{6.29}$$

□

7. Proof of Corollaries 1.5 and 1.6. Recall that ρ standard means that ρ is asymptotically periodic and regularly varying at infinity. The following lemma summarizes Corollaries 1.5 and 1.6.

LEMMA 7.1. *If either $m_\rho < \infty$ or ρ is standard with $m_\rho = \infty$, then $\liminf_{\beta \downarrow 0} h_c^{\mathrm{que}}(\beta)/\beta \ge K_c^*(\alpha)$ with*

$$K_c^*(\alpha) = \begin{cases} \dfrac{B(\alpha)}{\alpha}, & \text{for } 1 < \alpha < 2, \\ \dfrac{1+\alpha}{2\alpha}, & \text{for } \alpha \ge 2. \end{cases} \tag{7.1}$$

PROOF. The proof comes in 6 steps. In steps 1–3, we give the proof for the case where the disorder ω is standard Gaussian and the excursion length distribution ρ satisfies $\rho(k) \sim Ak^{-\alpha}$ as $k \to \infty$ for some $0 < A < \infty$ and $1 < \alpha < 2$. In steps 4–6, we explain how to extend the proof to arbitrary ω and ρ satisfying the stated conditions.

1. Our starting point is (6.22) with

$$\begin{aligned} \mathcal{N}(\beta, h) &= \sum_{m \in \mathbb{N}} \rho(m) \int_{\mathbb{R}^m} \nu(dx_1) \times \cdots \times \nu(dx_m) \\ &\qquad \times \left\{ \frac{1}{2}\big(1 + e^{-2\beta h m - 2\beta(x_1 + \cdots + x_m)}\big) \right\}^{1/\alpha} \\ &= \sum_{m \in \mathbb{N}} \rho(m) \int_{l \in \mathbb{R}} \nu^{\circledast m}(dl) \left\{ \frac{1}{2}\big(1 + e^{-2\beta h m - 2\beta l}\big) \right\}^{1/\alpha}, \end{aligned} \tag{7.2}$$

where $\nu^{\circledast m}$ is a m-fold convolution of ν. Pick $h = B\beta/\alpha$ with $B \ge 1$, introduce the variables

$$x = l/\sqrt{m}, \qquad y = (\beta/\alpha)^2 m \tag{7.3}$$

and write out

$$
\begin{aligned}
&\mathcal{N}(\beta, B\beta/\alpha) \\
&\quad = \sum_{m\in\mathbb{N}} \rho(m) \int_{l\in\mathbb{R}} N(0,m)(dl) \left\{\frac{1}{2}\big(1+e^{-2B(\beta^2/\alpha)m-2\beta l}\big)\right\}^{1/\alpha} \\
&\quad = \sum_{y\in(\beta/\alpha)^2\mathbb{N}} \rho\big(y(\alpha/\beta)^2\big) \int_{x\in\mathbb{R}} N(0,1)(dx) \left\{\frac{1}{2}\big(1+e^{-\alpha[2By+2\sqrt{y}x]}\big)\right\}^{1/\alpha} \\
&\quad = \sum_{y\in(\beta/\alpha)^2\mathbb{N}} \rho\big(y(\alpha/\beta)^2\big) E\big(f_\alpha(Z_{y,B})\big),
\end{aligned}
\tag{7.4}
$$

where $N(0,k)$ is the Gaussian distribution with mean 0 and variance k, f_α is the function defined in (6.24), and $Z_{y,B}$ is the random variable

$$
Z_{y,B} = e^{-2By-2\sqrt{y}X} \qquad \text{with } X \text{ standard Gaussian,} \tag{7.5}
$$

whose law we denote by P. Substract 1 to obtain

$$
\mathcal{N}(\beta, B\beta/\alpha) - 1 = \sum_{y\in(\beta/\alpha)^2\mathbb{N}} \rho\big(y(\alpha/\beta)^2\big)\big[E\big(f_\alpha(Z_{y,B})\big)-1\big]. \tag{7.6}
$$

2. Suppose that $\rho(m) \sim Am^{-\alpha}$ as $m\to\infty$ for some $0<A<\infty$ and $1<\alpha<2$. Then, letting $\beta\downarrow 0$ in (7.6), we obtain

$$
\begin{aligned}
&\lim_{\beta\downarrow 0} \frac{1}{\beta^{2(\alpha-1)}}\big[\mathcal{N}(\beta, B\beta/\alpha)-1\big] \\
&\qquad = \frac{A}{\alpha^{2(\alpha-1)}} \int_0^\infty dy\, y^{-\alpha}\big[E\big(f_\alpha(Z_{y,B})\big)-1\big].
\end{aligned}
\tag{7.7}
$$

Here, we note that the integral converges near $y=0$ because $\alpha<2$ and $E(f_\alpha(Z_{y,B}))-1 = O(y)$ as $y\downarrow 0$ [see (7.12) below], and also converges near $y=\infty$ because $\alpha>1$ and $E(f_\alpha(Z_{y,B})) \le (\frac{1}{2})^{1/\alpha}(1+E(Z_{y,B})) \le (\frac{1}{2})^{1/\alpha}(1+E(Z_{y,1})) = 2^{(\alpha-1)/\alpha} < \infty$ [recall that $f_\alpha(z) \le (\frac{1}{2})^{-1/\alpha}(1+z)$ and $B\ge 1$]. Next, abbreviate

$$
I_\alpha(B) = \int_0^\infty dy\, y^{-\alpha}\big[E\big(f_\alpha(Z_{y,B})\big)-1\big]. \tag{7.8}
$$

If $B=1$, then the integrand is strictly positive, because $z\mapsto f_\alpha(z)$ is strictly convex and $E(Z_{y,1})=1$ for all y, so that $E(f_\alpha(Z_{y,1})) > f_\alpha(E(Z_{y,1})) = f_\alpha(1) = 1$ by Jensen's inequality. Thus, we have $I_\alpha(1)>0$. However, $B\mapsto I_\alpha(B)$ is strictly decreasing and continuous on $[1,\infty)$, because $z\mapsto f_\alpha(z)$ is strictly increasing and continuous on $[0,\infty)$. Hence, there exists a $B(\alpha)>1$ such that $I_\alpha(B(\alpha))=0$ (use that $\lim_{B\to\infty} I_\alpha(B) = \int_0^\infty dy\, y^{-\alpha}[(\frac{1}{2})^{1/\alpha}-1] < 0$).

3. The estimate in step 2 implies that $\mathcal{N}(\beta, B\beta/\alpha) > 1$ for any $B \in (1, B(\alpha))$ and β small enough. Since $h \mapsto S_*^{\text{que}}(\beta, h)$ is nonincreasing and $S_*^{\text{que}}(\beta, h_c^{\text{que}}(\beta)) = 0$, it follows from (6.22) that $h_c^{\text{que}}(\beta) \geq B\beta/\alpha$ whenever $B \in (1, B(\alpha))$ and β small enough. These result in

$$\liminf_{\beta \downarrow 0} h_c^{\text{que}}(\beta)/\beta \geq \frac{B(\alpha)}{\alpha}. \tag{7.9}$$

4. If the disorder is not standard Gaussian, then the *same scaling* as in (7.7) holds because the disorder satisfies the central limit theorem (recall that we have assumed that the disorder has zero mean and unit variance). The finiteness of the moment generating function assumed in (1.2) suffices to justify this claim [in fact, all that is needed is (7.12) below]. If the excursion length distribution is modulated by a slowly varying function L, as in (1.14), then we can use the fact that $L(y(\alpha/\beta)^2) \sim L(1/\beta^2)$ as $\beta \downarrow 0$ uniformly in y on compact subsets of $(0, \infty)$ (Bingham, Goldie and Teugels [2], Theorem 1.2.1), and all that changes is that the left-hand side of (7.7) must be divided by an extra factor $L(1/\beta^2)$. We need that ρ is asymptotically periodic in order to get the integral over y w.r.t. the Lebesgue measure dy (modulo a factor 1 over the period, which comes in front and, therefore, is irrelevant).

5. We next turn to the case $\alpha \geq 2$. For $y \downarrow 0$,

$$\begin{aligned} Z_{y,B} - 1 &= e^{-2By - 2\sqrt{y}X} - 1 \\ &= \sqrt{y}(-2X) + y\bigl(-2B + 2X^2\bigr) + O\bigl(y^{3/2}\bigr), \end{aligned} \tag{7.10}$$

while for $z \to 1$,

$$f_\alpha(z) = 1 + \tfrac{1}{2}(z-1) + \tfrac{1}{8}(\alpha - 1)(z-1)^2 + O\bigl((z-1)^3\bigr). \tag{7.11}$$

Combining these expansions with the observation that X has zero mean and unit variance, we find that for $y \downarrow 0$,

$$E\bigl(f_\alpha(Z_{y,B})\bigr) = 1 + y\bigl[\tfrac{1}{2}(1+\alpha) - B\bigr] + O\bigl(y^{3/2}\bigr). \tag{7.12}$$

Since $E(f_\alpha(Z_{y,B}))$ is bounded from above, it follows from (7.12) that if $B < \frac{1}{2}(1+\alpha)$ and

$$\lim_{\beta \downarrow 0} \frac{\sum_{y \in (\beta/\alpha)^2\mathbb{N}, y > \epsilon} \rho(y(\alpha/\beta)^2)}{\sum_{y \in (\beta/\alpha)^2\mathbb{N}, y \leq \epsilon} y\rho(y(\alpha/\beta)^2)} = 0 \qquad \forall \epsilon > 0, \tag{7.13}$$

then the behavior of the sum in (7.6) for $\beta \downarrow 0$ is dominated by the small values of y, that is,

$$\begin{aligned} &\mathcal{N}(\beta, B\beta/\alpha) - 1 \\ &\quad \sim \left[\frac{1}{2}(1+\alpha) - B\right] \sum_{y \in (\beta/\alpha)^2\mathbb{N}, y \leq \epsilon} \bigl[y + O\bigl(y^{3/2}\bigr)\bigr]\rho\bigl(y(\alpha/\beta)^2\bigr) \\ &\quad = \left[\frac{1}{2}(1+\alpha) - B\right](\beta/\alpha)^2 \sum_{m \in \mathbb{N}, m \leq \epsilon(\alpha/\beta)^2} m\rho(m) \qquad \forall \epsilon > 0. \end{aligned} \tag{7.14}$$

The condition in (7.13) is equivalent to

$$\lim_{M\to\infty} \frac{M\sum_{m>M}\rho(m)}{\sum_{1\le m\le M} m\rho(m)} = 0. \tag{7.15}$$

Clearly, the condition in (7.15) is satisfied when $m_\rho = \sum_{m\in\mathbb{N}} m\rho(m) < \infty$ (because the numerator tends to zero and the denominator tends to m_ρ), in which case (7.14) yields

$$\lim_{\beta\downarrow 0} \frac{1}{\beta^2}[\mathcal{N}(\beta, B\beta/\alpha) - 1] = \left[\frac{1}{2}(1+\alpha) - B\right]\frac{1}{\alpha^2} m_\rho. \tag{7.16}$$

As in step 3, it therefore follows that $h_c^{\mathrm{que}}(\beta) \ge B\beta/\alpha$ for any $B < \frac{1}{2}(1+\alpha)$ and β small enough, which yields

$$\liminf_{\beta\downarrow 0} h_c^{\mathrm{que}}(\beta)/\beta \ge \frac{1+\alpha}{2\alpha}. \tag{7.17}$$

It remains to check (7.15) when $m_\rho = \infty$ and ρ is regularly varying at infinity with exponent $\alpha = 2$, that is, $\rho(m) = m^{-2}L(m)$ along the (asymptotically periodic) support of ρ with L slowly varying at infinity and not decaying too fast. Now, by [2], Theorem 1.5.10, we have $\sum_{m>M}\rho(m) = \sum_{m>M} m^{-2}L(m) \sim M^{-1}L(M)$, and so the numerator of (7.15) is $\sim L(M)$. On the other hand, by [2], Proposition 1.5.9a, $\lim_{M\to\infty} L(M)^{-1}\sum_{1\le m\le M} m^{-1}L(m) = \infty$, and so (7.15) indeed holds.

6. As in step 4, the argument in step 5 extends to arbitrary disorder subject to (1.2). □

8. Proof of Corollaries 1.7 and 1.8. Corollaries 1.7 and 1.8 are proved in Sections 8.1 and 8.2, respectively.

8.1. *Proof of Corollary* 1.7.

PROOF. Fix $(\beta, h) \in \mathrm{int}(\overline{\mathcal{D}}^{\mathrm{que}})$. We know that $\overline{S}^{\mathrm{que}}(\beta, h; 0) < 0$ (recall Figure 7) and $\sum_{n\in\mathbb{N}} \widetilde{Z}_n^{\beta,h,\omega} < \infty$. It follows from (3.16) that for every $\epsilon > 0$ and ω-a.s. there exists an $N_0 = N_0(\omega, \epsilon) < \infty$ such that

$$F_N^{\beta,h,\omega}(0) \le e^{N[\overline{S}^{\mathrm{que}}(\beta,h;0)+\epsilon]}, \qquad N \ge N_0. \tag{8.1}$$

For E an arbitrary event, write $\widetilde{Z}_n^{\beta,h,\omega}(E)$ to denote the constrained partition restricted to E. Estimate, for $M \in \mathbb{N}$ and ϵ small enough such that $\overline{S}^{\mathrm{que}}(\beta, h; 0) + \epsilon < 0$,

$$\begin{aligned} &\widetilde{\mathcal{P}}_n^{\beta,h,\omega}(\mathcal{M}_n \ge M) \\ &\quad = \frac{\widetilde{Z}_n^{\beta,h,\omega}(\mathcal{M}_n \ge M)}{\widetilde{Z}_n^{\beta,h,\omega}} \le \frac{\sum_{n\in\mathbb{N}} \widetilde{Z}_n^{\beta,h,\omega}(\mathcal{M}_n \ge M)}{\widetilde{Z}_n^{\beta,h,\omega}} \\ &\quad = \frac{1}{\widetilde{Z}_n^{\beta,h,\omega}} \sum_{N\ge M} F_N^{\beta,h,\omega}(0) \le \frac{2}{\rho(n)} \frac{e^{M[\overline{S}^{\mathrm{que}}(\beta,h;0)+\epsilon]}}{1 - e^{[\overline{S}^{\mathrm{que}}(\beta,h;0)+\epsilon]}}, \end{aligned} \tag{8.2}$$

where the second equality follows from (3.11)–(3.13). The second inequality follows from (8.1) and the bound $\widetilde{Z}_n^{\beta,h,\omega} \geq \frac{1}{2}\rho(n)$, the latter being immediate from (1.16) and the fact that every excursion has probability $\frac{1}{2}$ of lying below the interface. Since $\rho(n) = n^{-\alpha+o(1)}$, we get the claim by choosing $M = \lceil c \log n \rceil$ with c such that $\alpha + c[\overline{S}^{\mathrm{que}}(\beta, h; 0) + \epsilon] < 0$, and letting $n \to \infty$ followed by $\epsilon \downarrow 0$. $\square$

8.2. *Proof of Corollary* 1.8.

PROOF. Fix $(\beta, h) \in \mathcal{L}^{\mathrm{que}}$. We know that $g^{\mathrm{que}}(\beta, h) > 0$ and $S^{\mathrm{que}}(\beta, h; g^{\mathrm{que}}(\beta, h)) = 0$. It follows from (3.16) and (3.20) that for every $\epsilon, \delta > 0$ and ω-a.s. there exist $n_0 = n_0(\omega, \epsilon) < \infty$ and $M_0 = M_0(\omega, \delta) < \infty$ such that

$$\widetilde{Z}_n^{\beta,h,\omega} \geq e^{n[g^{\mathrm{que}}(\beta,h)-\epsilon]}, \qquad n \geq n_0,$$

$$F_M^{\beta,h,\omega}\big(g^{\mathrm{que}}(\beta, h) + \delta\big) \leq e^{M[S^{\mathrm{que}}(\beta,h;g^{\mathrm{que}}(\beta,h)+\delta)+\delta^2]}, \qquad M \geq M_0, \tag{8.3}$$

$$F_M^{\beta,h,\omega}\big(g^{\mathrm{que}}(\beta, h) - \delta\big) \leq e^{M[S^{\mathrm{que}}(\beta,h;g^{\mathrm{que}}(\beta,h)-\delta)+\delta^2]}, \qquad M \geq M_0.$$

For every $M_1, M_2 \in \mathbb{N}$ with $M_1 < M_2$, we have

$$\begin{aligned} &\widetilde{\mathcal{P}}_n^{\beta,h,\omega}(M_1 < \mathcal{M}_n < M_2) \\ &\quad = 1 - \big[\widetilde{\mathcal{P}}_n^{\beta,h,\omega}(\mathcal{M}_n \geq M_2) + \widetilde{\mathcal{P}}_n^{\beta,h,\omega}(\mathcal{M}_n \leq M_1)\big]. \end{aligned} \tag{8.4}$$

Below we show that the probabilities in the right-hand side of (8.4) vanish as $n \to \infty$ when $M_1 = \lceil c_1 n \rceil$ with $c_1 < C_-$ and $M_2 = \lceil c_2 n \rceil$ with $c_2 > C_+$, respectively, where

$$\begin{aligned} -\frac{1}{C_-} &= \left(\frac{\partial}{\partial g}\right)^- S^{\mathrm{que}}\big(\beta, h; g^{\mathrm{que}}(\beta, h)\big), \\ -\frac{1}{C_+} &= \left(\frac{\partial}{\partial g}\right)^+ S^{\mathrm{que}}\big(\beta, h; g^{\mathrm{que}}(\beta, h)\big), \end{aligned} \tag{8.5}$$

are the left-derivative and right-derivative of $g \mapsto S^{\mathrm{que}}(\beta, h; g)$ at $g = g^{\mathrm{que}}(\beta, h)$, which exist by convexity, are strictly negative (recall Figure 6) and satisfy $C_- \leq C_+$. Throughout the proof, we assume that $M_1 \geq M_0$.

1. Put $M_2 = \lceil c_2 n \rceil$, and abbreviate

$$a(\beta, h, \delta) = S^{\mathrm{que}}\big(\beta, h; g^{\mathrm{que}}(\beta, h) + \delta\big) + \delta^2, \tag{8.6}$$

where we choose δ small enough such that $a(\beta, h, \delta) < 0$ [recall Figure 6(1)]. Estimate

$$\begin{aligned} \widetilde{\mathcal{P}}_n^{\beta,h,\omega}(\mathcal{M}_n \geq M_2) &= \frac{\widetilde{Z}_n^{\beta,h,\omega}(\mathcal{M}_n \geq M_2)}{\widetilde{Z}_n^{\beta,h,\omega}} \\ &\leq e^{n[\epsilon+\delta]}\widetilde{Z}_n^{\beta,h,\omega}(\mathcal{M}_n \geq M_2)e^{-n[g^{\mathrm{que}}(\beta,h)+\delta]} \end{aligned} \tag{8.7}$$

$$\begin{aligned}
&\le e^{n[\epsilon+\delta]} \sum_{n' \in \mathbb{N}} \widetilde{Z}_{n'}^{\beta,h,\omega}(\mathcal{M}_{n'} \ge M_2) e^{-n'[g^{\text{que}}(\beta,h)+\delta]} \\
&= e^{n[\epsilon+\delta]} \sum_{N \ge M_2} F_N^{\beta,h,\omega}\bigl(g^{\text{que}}(\beta,h)+\delta\bigr) \\
&\le e^{n[\epsilon+\delta]} \sum_{N \ge M_2} e^{Na(\beta,h,\delta)} \\
&= \frac{e^{n[\epsilon+\delta+c_2 a(\beta,h,\delta)]}}{1-e^{a(\beta,h,\delta)}}.
\end{aligned}$$

The first inequality follows from the first line in (8.3), the second equality from (3.11)–(3.13), and the third inequality from (8.3). The claim follows by choosing $\epsilon > 0$, choosing c_2 such that

$$\epsilon + \delta + c_2 a(\beta,h,\delta) < 0, \tag{8.8}$$

letting $n \to \infty$ followed by $\epsilon \downarrow 0$ and $\delta \downarrow 0$, and using that

$$\lim_{\delta \downarrow 0} \frac{1}{\delta} a(\beta,h,\delta) = \left(\frac{\partial}{\partial g}\right)^{+} S^{\text{que}}\bigl(\beta,h; g^{\text{que}}(\beta,h)\bigr) = -\frac{1}{C_+}. \tag{8.9}$$

2. Put $M_1 = \lceil c_1 n \rceil$ and abbreviate

$$b(\beta,h,\delta) = S^{\text{que}}\bigl(\beta,h; g^{\text{que}}(\beta,h)-\delta\bigr) + \delta^2, \tag{8.10}$$

where we choose δ small enough such that $b(\beta,h,\delta) > 0$. Split

$$\widetilde{\mathcal{P}}_n^{\beta,h,\omega}(\mathcal{M}_n \le M_1) = I + II \tag{8.11}$$

with

$$\begin{aligned}
I &= \frac{\widetilde{Z}_n^{\beta,h,\omega}(\mathcal{M}_n < M_0)}{\widetilde{Z}_n^{\beta,h,\omega}}, \\
II &= \frac{\widetilde{Z}_n^{\beta,h,\omega}(M_0 \le \mathcal{M}_n \le M_1)}{\widetilde{Z}_n^{\beta,h,\omega}}.
\end{aligned} \tag{8.12}$$

Since

$$\begin{aligned}
I &\le \exp\bigl[-n\bigl(g^{\text{que}}(\beta,h)-\epsilon\bigr)\bigr] \widetilde{Z}_n^{\beta,h,\omega}(\mathcal{M}_n < M_0) \\
&\le \exp\left[-\frac{1}{2} n\bigl(g^{\text{que}}(\beta,h)-\epsilon\bigr)\right] \sum_{N < M_0} F_N^{\beta,h,\omega}\left(\frac{1}{2}\bigl(g^{\text{que}}(\beta,h)-\epsilon\bigr)\right) \\
&\le \exp\left[-\frac{1}{2} n\bigl(g^{\text{que}}(\beta,h)-\epsilon\bigr)\right] M_0 \exp\left[M_0 K\left(\omega, \frac{1}{2}\bigl(g^{\text{que}}(\beta,h)-\epsilon\bigr)\right)\right],
\end{aligned} \tag{8.13}$$

this term is harmless as $n \to \infty$ [recall (3.27)]. Repeat the arguments leading to (8.7), to estimate

$$
\begin{aligned}
II &\le e^{n[\epsilon-\delta]} \sum_{n' \in \mathbb{N}} \widetilde{Z}_{n'}^{\beta,h,\omega}(M_0 \le \mathcal{M}_{n'} \le M_1) e^{-n'[g^{\text{que}}(\beta,h)-\delta]} \\
&= e^{n[\epsilon-\delta]} \sum_{M_0 \le N \le M_1} F_N^{\beta,h,\omega}\big(g^{\text{que}}(\beta,h)-\delta\big) \\
&\le e^{n[\epsilon-\delta]} \sum_{M_0 \le N \le M_1} e^{Nb(\beta,h,\delta)} \\
&\le e^{n[\epsilon-\delta+c_1 b(\beta,h,\delta)]} \sum_{N \le M_1} e^{[N-M_1]b(\beta,h,\delta)} \\
&\le \frac{e^{n[\epsilon-\delta+c_1 b(\beta,h,\delta)]}}{1-e^{-b(\beta,h,\delta)}}.
\end{aligned} \tag{8.14}
$$

Therefore, the assertion follows by choosing $\epsilon > 0$, then choosing c_1 such that

$$\epsilon - \delta + c_1 b(\beta, h, \delta) < 0, \tag{8.15}$$

letting $n \to \infty$ followed by $\epsilon \downarrow 0$ and $\delta \downarrow 0$, and using that

$$\lim_{\delta \downarrow 0} \frac{1}{\delta} b(\beta, h, \delta) = -\left(\frac{\partial}{\partial g}\right)^{-} S^{\text{que}}\big(\beta, h; g^{\text{que}}(\beta, h)\big) = \frac{1}{C_-}. \tag{8.16}$$

Recalling (8.4), we have now proved that

$$\lim_{n\to\infty} \widetilde{\mathcal{P}}_n^{\beta,h,\omega}\big(\lceil c_1 n \rceil < \mathcal{M}_n < \lceil c_2 n \rceil\big) = 1 \qquad \forall c_1 < C_-, c_2 > C_+. \tag{8.17}$$

Finally, if (1.33) holds, then $C_- = C_+$, and we get the law of large numbers in (1.32). $\square$

APPENDIX A: CONTROL OF $\Phi_{\beta,h}$

In Appendix A.1, we prove the bound in (3.27) (Lemma A.1 below). In Appendix A.2, we prove that $h(\pi_1 Q | q_{\rho,\nu}) < \infty$ implies that $\Phi_{\beta,h}(Q) < \infty$ for all $\beta, h > 0$ (Lemma A.3 below). In both proofs, we make use of a concentration of measure estimate for the disorder ω whose proof is given in Appendix D.

A.1. Proof of ω-a.s. boundedness of $-gm_{R_N^\omega} + \Phi_{\beta,h}(R_N^\omega)$ for $g > 0$. Recall the definition of $\psi_{\beta,h}$ in (3.10) and R_N^ω in (3.14).

LEMMA A.1. *Fix $\beta, h, g > 0$. Then ω-a.s. there exists a $K(\omega, g) < \infty$ such that, for all $N \in \mathbb{N}$ and for all sequences $0 = k_0 < k_1 < \cdots < k_N < \infty$,*

$$-gk_N + \sum_{i=1}^{N} \log \psi_{\beta,h}^{\omega}\big((k_{i-1}, k_i]\big) \le K(\omega, g)N. \tag{A.1}$$

PROOF. The proof comes in 6 steps.

1. For $m \in \mathbb{N}$, let

$$\mathcal{J}_m^\omega = \Big\{ J \subset \mathbb{N} \text{ finite interval} : m \le -2\beta \sum_{k \in J} (\omega_k + h) < m+1 \Big\}. \tag{A.2}$$

For $l, m, n \in \mathbb{N}$, let

$$\begin{aligned} &R^\omega(l, m, n) \\ &\quad = \text{number of intervals in } \mathcal{J}_m^\omega \text{ of length } l \text{ whose endpoints are } \le n \end{aligned} \tag{A.3}$$

and put $R^\omega(m,n) = \sum_{l=1}^n R^\omega(l,m,n)$. Define

$$\begin{aligned} A(m,j) &= \{\omega : R^\omega(m, jm^4) > j\}, \qquad j, m \in \mathbb{N}, \\ A(m) &= \bigcup_{j \in \mathbb{N}} A(m,j). \end{aligned} \tag{A.4}$$

Below we show that

$$\sum_{m \in \mathbb{N}} \mathbb{P}(A(m)) < \infty. \tag{A.5}$$

Hence, by the Borel–Cantelli lemma, ω-a.s. there exists an $M(\omega) \in \mathbb{N}$ such that $\omega \notin A(m)$ for all $m \ge M(\omega)$.

2. We first show that (A.5) implies (A.1). Let $0 = k_0 < k_1 < \cdots < k_N < \infty$ be an arbitrary sequence and consider the intervals $I_j = (k_{j-1}, k_j]$, $j = 1, \dots, N$. If $I_j \in \mathcal{J}_m^\omega$ with $m < M(\omega)$, then $\log \psi_{\beta,h}^\omega(I_j) \le \log[\frac{1}{2}(1 + e^{M(\omega)})] \le M(\omega)$ and, therefore,

$$\sum_{m=0}^{M(\omega)-1} \sum_{\substack{1 \le j \le N \\ I_j \in \mathcal{J}_m^\omega}} \log \psi_{\beta,h}^\omega(I_j) \le M(\omega) N. \tag{A.6}$$

To deal with the remaining intervals, let

$$r^\omega(m) = \big|\{1 \le j \le N : I_j \in \mathcal{J}_m^\omega\}\big|, \qquad m \ge M(\omega). \tag{A.7}$$

If $m \ge M(\omega)$, then $\omega \notin A(m)$ and it follows from (A.4) that $R(m, jm^4) \le j$ for all $j \in \mathbb{N}$, which implies that $k_N \ge r^\omega(m) m^4$. Therefore,

$$\begin{aligned} k_N &\ge \max_{m \ge M(\omega)} r^\omega(m) m^4 \\ &\ge \sum_{m \ge M(\omega)} r^\omega(m) m^4 \frac{m^{-2}}{\sum_{n \ge M(\omega)} n^{-2}} \ge C \sum_{m \ge M(\omega)} r^\omega(m) m^2 \end{aligned} \tag{A.8}$$

with $C = 6/\pi^2$. Combining (A.6) and (A.8), we obtain

$$
\begin{aligned}
&-gk_N + \sum_{i=1}^{N} \log \psi_{\beta,h}^{\omega}\big((k_{i-1}, k_i]\big) \\
&\qquad \le M(\omega)N + \sum_{m \ge M(\omega)} r^{\omega}(m)\big[-gCm^2 + (m+1)\big],
\end{aligned}
\tag{A.9}
$$

where we use that $\log \psi_{\beta,h}^{\omega} < m + 1$ on $\mathcal{J}_m^{\omega}$. Since $g > 0$, we have

$$
\max_{m \in \mathbb{N}} \big[-gCm^2 + (m+1)\big] = C(g) \le 1 + (1/4gC) < \infty.
$$

Since $\sum_{m \in \mathbb{N}} r^{\omega}(m) \le N$, the claim in (A.1) follows with $K(\omega, g) = M(\omega) + C(g)$. Thus, it remains to prove (A.5).

3. By the concentration of measure estimate in Lemma D.1 with $n = l$, $A = \frac{m}{2\beta}$ and $B = h$, there exists a $C = C(\beta, h) > 0$ such that, for all m large enough and all $l \in \mathbb{N}$,

$$
\mathbb{P}\left(-\beta \sum_{k=1}^{l} (\omega_k + h) \ge m\right) \le e^{-C(l+m)}.
\tag{A.10}
$$

The constant C remains fixed for the rest of the proof. For $j \in \mathbb{N}$, let $L(j) = \lceil 3(\log j)/C \rceil$. For each $j \in \mathbb{N}$, we will choose a sequence $k_j(1), \dots, k_j(L(j) - 1)$ such that

$$
\sum_{l=1}^{L(j)-1} k_j(l) \le \frac{1}{2} j, \qquad j \in \mathbb{N}.
\tag{A.11}
$$

Consider the events

$$
D(l, m, j) = \begin{cases} \{\omega : R^{\omega}(l, m, jm^4) > k_j(l)\}, & 1 \le l \le L(j) - 1, \\ \displaystyle\bigcup_{l' \ge L(j)} \{\omega : R^{\omega}(l', m, jm^4) \ne 0\}, & l = L(j). \end{cases}
\tag{A.12}
$$

By (A.4), condition (A.11) implies

$$
A(m, j) \subset \bigcup_{l=1}^{L(j)} D(l, m, j)
\tag{A.13}
$$

and, therefore,

$$
\sum_{m \in \mathbb{N}} \mathbb{P}\big(A(m)\big) \le \sum_{m \in \mathbb{N}} \sum_{j \in \mathbb{N}} \sum_{l=1}^{L(j)} \mathbb{P}\big(D(l, m, j)\big).
\tag{A.14}
$$

4. In order to estimate the right-hand side of (A.14), we recursively define for each $l \in \mathbb{N}$ the sequence of random times

$$T_0 = 0, \qquad T_{i+1} = \inf\{n \geq T_i : (n, n+l] \in \mathcal{J}_m^\omega\} + l, \qquad i \in \mathbb{N} \tag{A.15}$$

and the sequence of random times $(\tau_i)_{i \in \mathbb{N}}$ by putting $\tau_i = T_i - T_{i-1}$, $i \in \mathbb{N}$. The τ_i's are i.i.d. To emphasize that T_i and τ_i depend on l and m, we write $T_i(l,m)$ and $\tau_i(l,m)$. Note that for $n \geq l$ there are $n-l+1$ possibilities to place an interval of length l inside $(0,n] \cap \mathbb{N}$. Therefore, by the union bound, we have

$$\mathbb{P}\big(\tau_1(l,m) \leq n\big) \leq (n-l+1)\mathbb{P}\left(-\beta \sum_{k=1}^{l}(\omega_k + h) \geq m\right) \leq n e^{-C(l+m)}, \tag{A.16}$$

where we use (A.2) and (A.10). We first estimate $\mathbb{P}(D(L(j), m, j))$. If $\omega \in D(L(j), m, j)$, then by (A.3) and (A.12) there exists an $l \geq L(j)$ with $\tau_1(l,m) \leq jm^4$. Therefore, by (A.16) with $n = jm^4$, we have

$$\begin{aligned}
&\sum_{m \in \mathbb{N}} \sum_{j \in \mathbb{N}} \mathbb{P}\big(D\big(L(j), m, j\big)\big) \\
&\quad \leq \sum_{m \in \mathbb{N}} \sum_{j \in \mathbb{N}} \sum_{\substack{l \in \mathbb{N} \\ l \geq L(j)}} \mathbb{P}\big(\tau_1(l,m) \leq jm^4\big) \\
&\quad \leq \left(\sum_{m \in \mathbb{N}} m^4 e^{-Cm}\right)\left(\sum_{j \in \mathbb{N}} j \sum_{l \geq L(j)} e^{-Cl}\right) < \infty,
\end{aligned} \tag{A.17}$$

where the last sum is finite by our choice of $L(j)$. It therefore remains to estimate $\mathbb{P}(D(l,m,j))$ for $1 \leq l < L(j)$.

5. We now specify the sequence $k_j(l)$ of (A.11) by setting

$$k_j(l) = c_1 j e^{-al}, \tag{A.18}$$

where $0 < a < C/2$ is taken small enough so that $je^{-aL(j)} \geq j^{1/2}$ for all $j \in \mathbb{N}$, and $c_1 > 0$ is taken small enough (depending on a) so that (A.11) is satisfied. Since $j \mapsto L(j)/j^{1/4}$ is bounded from above we have, for all $1 \leq l < L(j)$,

$$\left\lceil \frac{k_j(l)}{l} \right\rceil \geq \frac{k_j(l)}{l} = c_1 \frac{je^{-al}}{l} \geq c_1 \frac{je^{-aL(j)}}{L(j)} \geq c_1 \frac{j^{1/2}}{L(j)} \geq c_2 j^{1/4} \tag{A.19}$$

for some $c_2 > 0$. If $\omega \in D(l,m,j)$ for $1 \leq l < L(j)$, then $T_{\lceil k_j(l)/l \rceil}(l,m) \leq jm^4$ by (A.12). [The fact that we take index $\lceil k_j(l)/l \rceil$ instead of $k_j(l)$ is due to the possible overlaps of the intervals contributing to the event $D(l,m,j)$.] Therefore

$$D(l,m,j) \subset \{T_{\lceil k_j(l)/l \rceil}(l,m) \leq jm^4\} = \left\{\sum_{i=1}^{\lceil k_j(l)/l \rceil} \tau_i(l,m) \leq jm^4\right\}, \tag{A.20}$$

so that, with the help of Lemma A.2 below [with $k = \lceil k_j(l)/l \rceil$, $n = jm^4$ and $\varepsilon = e^{-C(l+m)}$], we get

$$\mathbb{P}\big(D(l,m,j)\big) \le \exp\left[\frac{1}{2}\lceil k_j(l)/l \rceil \log\left(\frac{jm^4 e^{-C(l+m)}}{\lceil k_j(l)/l \rceil} + e^{-C(l+m)}\right)\right]. \tag{A.21}$$

(Note that the conditions in Lemma A.2 are satisfied for m large enough.) Estimate

$$\sum_{m\in\mathbb{N}} \sum_{j\in\mathbb{N}} \sum_{1\le l<L(j)} \mathbb{P}\big(D(l,m,j)\big) \le \sum_{m\in\mathbb{N}} \sum_{j\in\mathbb{N}} L(j) \max_{1\le l<L(j)} \mathbb{P}\big(D(l,m,j)\big) \tag{A.22}$$

and note that, by (A.18), we have

$$\frac{jm^4 e^{-C(l+m)}}{\lceil k_j(l)/l \rceil} \le \frac{m^4 l e^{-C(l+m)+al}}{c_1} \le \frac{1}{c_1} m^4 e^{-Cm} l e^{-Cl/2}. \tag{A.23}$$

We now choose an M so large that

$$\frac{1}{c_1} m^4 e^{-Cm} l e^{-Cl/2} + e^{-C(l+m)} \le e^{-Cm/2}, \qquad l \in \mathbb{N}, m > M. \tag{A.24}$$

Then, by (A.19) and (A.23)–(A.24), the right-hand side of (A.21) is bounded from above by

$$\exp\left[-\frac{1}{2} c_2 C j^{1/4} m\right] \tag{A.25}$$

and since

$$\sum_{j\in\mathbb{N}} \sum_{m\ge M} L(j) \exp\left[-\frac{1}{2} c_2 C j^{1/4} m\right] < \infty, \tag{A.26}$$

we have from (A.21)–(A.22) that

$$\sum_{m\ge M} \sum_{j\in\mathbb{N}} \sum_{1\le l<L(j)} \mathbb{P}\big(D(l,m,j)\big) < \infty. \tag{A.27}$$

Together with (A.14) and (A.17), this proves (A.5).

6. It remains to prove the following lemma.

LEMMA A.2. *Let τ be an $\mathbb{N}$-valued random variable such that $P(\tau \le n) \le \varepsilon n$ for all $n \in \mathbb{N}$ and some $0 < \varepsilon \le \frac{1}{4}$. Let τ_i, $i \in \mathbb{N}$, be i.i.d. copies of τ. If $k, n \in \mathbb{N}$ satisfy $k \ge 10\varepsilon n$, then*

$$P\left(\sum_{i=1}^{k} \tau_i \le n\right) \le \exp\left[\frac{1}{2} k \log\left(\frac{\varepsilon n}{k} + \varepsilon\right)\right]. \tag{A.28}$$

PROOF. Estimate, for $\lambda > 0$,

$$P\left(\sum_{i=1}^{k}\tau_i \le n\right) = P\left(e^{-\lambda\sum_{i=1}^{k}\tau_i} \ge e^{-\lambda n}\right) \le e^{\lambda n}\left[E\left(e^{-\lambda\tau}\right)\right]^k. \tag{A.29}$$

We have (for ease of notation we pretend that $1/\varepsilon$ is integer)

$$\begin{aligned}
\frac{E(e^{-\lambda\tau})}{1-e^{-\lambda}} &= \sum_{n\in\mathbb{N}} e^{-\lambda n} P(\tau \le n) \\
&\le \varepsilon \sum_{n=1}^{(1/\varepsilon)-1} n e^{-\lambda n} + \sum_{n=1/\varepsilon}^{\infty} e^{-\lambda n} \\
&= \varepsilon \sum_{k=1}^{(1/\varepsilon)-1} \sum_{n=k}^{(1/\varepsilon)-1} e^{-\lambda n} + \frac{e^{-\lambda/\varepsilon}}{1-e^{-\lambda}} \\
&= \frac{1}{1-e^{-\lambda}}\left(\varepsilon \sum_{k=1}^{(1/\varepsilon)-1}\left[e^{-\lambda k} - e^{-\lambda/\varepsilon}\right] + e^{-\lambda/\varepsilon}\right) \\
&= \frac{1}{1-e^{-\lambda}}\left(\varepsilon\left[\sum_{k=1}^{(1/\varepsilon)-1} e^{-\lambda k} - \sum_{k=1}^{(1/\varepsilon)-1} e^{-\lambda/\varepsilon}\right] + e^{-\lambda/\varepsilon}\right) \\
&= \frac{1}{1-e^{-\lambda}}\left(\left[\frac{\varepsilon(e^{-\lambda} - e^{-\lambda/\varepsilon})}{1-e^{-\lambda}} - (1-\varepsilon)e^{-\lambda/\varepsilon}\right] + e^{-\lambda/\varepsilon}\right) \\
&= \varepsilon\frac{e^{-\lambda}(1-e^{-\lambda/\varepsilon})}{(1-e^{-\lambda})^2} \le \varepsilon\frac{1}{(1-e^{-\lambda})^2}.
\end{aligned} \tag{A.30}$$

It follows from (A.30) that

$$E\left(e^{-\lambda\tau}\right) \le \varepsilon\frac{1}{1-e^{-\lambda}}, \qquad 0 < \lambda \le 1. \tag{A.31}$$

Writing $\lambda = \gamma\varepsilon$ with $0 < \gamma\varepsilon \le 1$, we thus have

$$E\left(e^{-\lambda\tau}\right) \le \varepsilon\frac{1}{1-e^{-\gamma\varepsilon}} \tag{A.32}$$

and hence, by (A.29),

$$\begin{aligned}
P\left(\sum_{i=1}^{k}\tau_i \le n\right) &\le \exp\left[\gamma\varepsilon n + k\log\left(\frac{\varepsilon}{1-e^{-\gamma\varepsilon}}\right)\right] \\
&= \exp\left[k\left(\delta\gamma + \log\left(\frac{\varepsilon}{1-e^{-\gamma\varepsilon}}\right)\right)\right],
\end{aligned} \tag{A.33}$$

where we put $\delta = \varepsilon n/k$. Choose

$$\gamma = \frac{1}{\varepsilon}\log\left(1+\frac{\varepsilon}{\delta}\right). \tag{A.34}$$

Then the factor multiplying k in the right-hand side of (A.33) is

$$\frac{\delta}{\varepsilon}\log\left(1+\frac{\varepsilon}{\delta}\right)+\log(\delta+\varepsilon) \le 1+\log(\delta+\varepsilon) \le \frac{1}{2}\log(\delta+\varepsilon), \tag{A.35}$$

where the last inequality holds when $\delta+\varepsilon \le 1/e^2 < 1/10$. Hence, (A.28) holds □

Steps 1–6 complete the proof of Lemma A.1. □

A.2. Condition for finiteness of $\Phi_{\beta,h}(Q)$. The following lemma will be needed in Appendix B.

LEMMA A.3. *Fix $\beta, h > 0$, $\rho \in \mathcal{P}(\mathbb{N})$ and $\nu \in \mathcal{P}(\mathbb{R})$. Then there are finite and positive constants γ and $K = K(\beta, h, \rho, \nu, \gamma)$ such that, for all $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{\mathbb{R}}^{\mathbb{N}})$ with $h(\pi_1 Q|q_{\rho,\nu}) < \infty$,*

$$\Phi_{\beta,h}(Q) \le \gamma h(\pi_1 Q|q_{\rho,\nu}) + K. \tag{A.36}$$

PROOF. Abbreviate

$$\begin{aligned} f(y) &= \frac{d(\pi_1 Q)}{dq_{\rho,\nu}}(y),\\ u(y) &= -2\beta[\tau(y)h+\sigma(y)], \qquad y \in \widetilde{\mathbb{R}} = \bigcup_{n\in\mathbb{N}} \mathbb{R}^n. \end{aligned} \tag{A.37}$$

For $\gamma > 0$ (later we will choose γ) and $n, m \in \mathbb{N}$, define

$$\begin{aligned} A_{m,n} &= \{y \in \mathbb{R}^n : m-1 \le \gamma \log f(y) < m\},\\ A_{0,n} &= \{y \in \mathbb{R}^n : 0 \le f(y) < 1\},\\ B_{m,n} &= \{y \in \mathbb{R}^n : m-1 \le u(y) < m\}. \end{aligned} \tag{A.38}$$

Note that

$$\mathbb{R}^n = A_{0,n} \cup \left[\bigcup_{m\in\mathbb{N}} A_{m,n}\right], \qquad n \in \mathbb{N} \tag{A.39}$$

and that

$$B_n = \bigcup_{m\in\mathbb{N}} B_{m,n}, \qquad n \in \mathbb{N}, \tag{A.40}$$

is the set of points $y \in \mathbb{R}^n$ for which $u(y) \geq 0$. This gives rise to the decomposition

$$\begin{aligned}
\Phi_{\beta,h}(Q) &= \sum_{n\in\mathbb{N}} \int_{\mathbb{R}^n} \log\biggl(\frac{1}{2}\bigl[1+e^{u(y)}\bigr]\biggr)(\pi_1 Q)(dy) \\
&\leq \sum_{n\in\mathbb{N}} \int_{\mathbb{R}^n} \log\bigl(1 \vee e^{u(y)}\bigr)(\pi_1 Q)(dy) \\
&= \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \int_{B_{m,n}} u(y) f(y) q_{\rho,\nu}(dy) \\
&= I + II + III
\end{aligned} \tag{A.41}$$

with

$$\begin{aligned}
I &= \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \int_{[\bigcup_{l\in\mathbb{N}_0} B_{m+l,n}]\cap A_{m,n}} u(y) f(y) q_{\rho,\nu}(dy), \\
II &= \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \int_{A_{m,n}\cap[\bigcup_{l=1}^{m-1} B_{l,n}]} u(y) f(y) q_{\rho,\nu}(dy), \\
III &= \sum_{n\in\mathbb{N}} \int_{A_{0,n}\cap[\bigcup_{m\in\mathbb{N}} B_{m,n}]} u(y) f(y) q_{\rho,\nu}(dy).
\end{aligned} \tag{A.42}$$

The terms I and II deal with the set $B_n \cap \bigcup_{m\in\mathbb{N}} A_{m,n}$, while III deals with the set $B_n \cap A_{0,n}$. Note that

$$\begin{aligned}
I &\leq \sum_{n\in\mathbb{N}} \rho(n) \sum_{m\in\mathbb{N}} e^{m/\gamma} \sum_{l\in\mathbb{N}_0} (m+l)\mathbb{P}(B_{m+l,n}), \\
III &\leq \sum_{n\in\mathbb{N}} \rho(n) \sum_{m\in\mathbb{N}} m\mathbb{P}(B_{m,n}),
\end{aligned} \tag{A.43}$$

where we recall that $\mathbb{P} = \nu^{\otimes\mathbb{N}}$. The upper bound on I uses that $f \leq e^{m/\gamma}$ on $A_{m,n}$ and $u < m$ on $B_{m,n}$. The upper bound on III uses that $f \leq 1$ on $A_{0,n}$ and $u < m$ on $B_{m,n}$. We need to show that each of the three terms is finite. Note that in (A.43) the bound on I exceeds the bound on III. Hence, it suffices to focus on I and II.

I: Estimate

$$\begin{aligned}
I &\leq \sum_{n\in\mathbb{N}} \rho(n) \sum_{m\in\mathbb{N}} e^{m/\gamma} \sum_{l\in\mathbb{N}_0} (m+l)\mathbb{P}(B_{m+l,n}) \\
&\leq \sum_{n\in\mathbb{N}} \rho(n) \sum_{m\in\mathbb{N}} e^{m/\gamma} \sum_{l\in\mathbb{N}_0} (l+m)\mathbb{P}\Biggl(\sum_{k=1}^{n} \omega_k \leq -\biggl[nh + \frac{l+m-1}{2\beta}\biggr]\Biggr) \\
&\leq \sum_{n\in\mathbb{N}} \rho(n) e^{-\overline{C}n} \sum_{m\in\mathbb{N}} e^{m/\gamma} \sum_{l\in\mathbb{N}_0} (l+m) \exp\biggl[-\frac{\overline{C}(l+m-1)}{2\beta}\biggr] < \infty,
\end{aligned} \tag{A.44}$$

where the third inequality follows from Lemma D.1, with $A = \frac{l+m-1}{2\beta}$, $B = h$ [$\overline{C} > 0$ depends on β, h; see (D.5)–(D.7)], and the sum is finite when $\gamma > 2\beta/\overline{C}$.

II: Use that $u(y) < m-1 \le \gamma \log f(y)$ for $y \in A_{m,n} \cap [\bigcup_{l=1}^{m-1} B_{l,n}]$, to estimate

$$
\begin{aligned}
II &\le \gamma \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \int_{A_{m,n}\cap[\bigcup_{l=1}^{m-1} B_{l,n}]} f(y) \log f(y) q_{\rho,\nu}(dy) \\
&\le \gamma \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \int_{A_{m,n}} f(y) \log f(y) q_{\rho,\nu}(dy) \\
&= \gamma \sum_{n\in\mathbb{N}} \int_{\mathbb{R}^n\setminus A_{0,n}} f(y) \log f(y) q_{\rho,\nu}(dy) < \infty.
\end{aligned}
\tag{A.45}
$$

The finiteness of the last term stems from the fact that

$$
\begin{aligned}
h(\pi_1 Q | q_{\rho,\nu}) = &\sum_{n\in\mathbb{N}} \int_{\mathbb{R}^n\setminus A_{0,n}} f(y) \log f(y) q_{\rho,\nu}(dy) \\
&+ \sum_{n\in\mathbb{N}} \int_{A_{0,n}} f(y) \log f(y) q_{\rho,\nu}(dy)
\end{aligned}
\tag{A.46}
$$

is assumed to be finite, while the second term in the right-hand side of (A.46) lies in $[-1/e, 0]$. □

APPENDIX B: APPLICATION OF VARADHAN'S LEMMA

This appendix settles (3.20) for $\beta, h > 0$ and $g > 0$.

LEMMA B.1. *For all $\beta, h > 0$ and $g > 0$,*

$$
\overline{S}^{\text{que}}(\beta, h; g) = \sup_{Q\in\mathcal{C}^{\text{fin}}\cap\mathcal{R}} [\Phi_{\beta,h}(Q) - g m_Q - I^{\text{ann}}(Q)],
\tag{B.1}
$$

where $\overline{S}^{\text{que}}(\beta, h; g)$ is the ω-a.s. constant limsup defined in (3.16).

PROOF. Throughout the proof, $\beta, h > 0$ and $g > 0$ are fixed. Note that, since $h(\pi_1 Q | q_{\rho,\nu}) \le H(Q | q_{\rho,\nu}^{\otimes\mathbb{N}}) = I^{\text{ann}}(Q) < \infty$, it follows from (3.6)–(3.7) and Lemma A.3 that $\Phi_{\beta,h}(Q)$ is finite on $\mathcal{C}^{\text{fin}} = \{Q \in \mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}}) : I^{\text{ann}}(Q) < \infty, m_Q < \infty\}$.

Lower bound: Because $\Phi_{\beta,h}$ is lower semicontinuous on $\mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}})$ (by Fatou) and finite on $\mathcal{C}^{\text{fin}}$, the set

$$
\mathcal{A}_\epsilon = \{Q' \in \mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}}) : \Phi_{\beta,h}(Q') > \Phi_{\beta,h}(Q) - \epsilon\}
\tag{B.2}
$$

is open for every $Q \in \mathcal{C}^{\mathrm{fin}}$ and $\epsilon > 0$. Fix $Q \in \mathcal{C}^{\mathrm{fin}} \cap \mathcal{R}$ and $\epsilon > 0$, and use (3.15)–(3.16) to estimate

$$
\begin{aligned}
\overline{S}^{\mathrm{que}}(\beta, h; g) &= \log \mathcal{N}(g) + \limsup_{N\to\infty} \frac{1}{N} \log E_g^*(e^{N\Phi_{\beta,h}(R_N^\omega)}) \\
&\geq \log \mathcal{N}(g) + \liminf_{N\to\infty} \frac{1}{N} \log E_g^*(e^{N\Phi_{\beta,h}(R_N^\omega)} 1_{\mathcal{A}_\epsilon}(R_N^\omega)) \\
&\geq \log \mathcal{N}(g) + \inf_{Q'\in\mathcal{A}_\epsilon} \Phi_{\beta,h}(Q') + \liminf_{N\to\infty} \frac{1}{N} \log P_g^*(\mathcal{A}_\epsilon) \\
&\geq \log \mathcal{N}(g) + \inf_{Q'\in\mathcal{A}_\epsilon} \Phi_{\beta,h}(Q') - \inf_{Q'\in\mathcal{A}_\epsilon} I_g^{\mathrm{que}}(Q') \\
&\geq \log \mathcal{N}(g) + \Phi_{\beta,h}(Q) - I_g^{\mathrm{que}}(Q) - \epsilon,
\end{aligned}
\tag{B.3}
$$

where in the third inequality we use the quenched LDP in Theorem 2.3. Next, note that $I_g^{\mathrm{que}}(Q) = I_g^{\mathrm{ann}}(Q)$ for $Q \in \mathcal{R}$ by Theorem 2.3 and $I_g^{\mathrm{ann}}(Q) = I^{\mathrm{ann}}(Q) + \log \mathcal{N}(g) + g m_Q$ for $Q \in \mathcal{C}^{\mathrm{fin}}$ by Lemma 2.1. Insert these identities, take the supremum over $Q \in \mathcal{C}^{\mathrm{fin}} \cap \mathcal{R}$ and let $\epsilon \downarrow 0$, to arrive at the desired lower bound.

Upper bound: The proof of the upper bound uses a truncation argument and comes in 4 steps.

1. Abbreviate $\chi(y) = \log \phi_{\beta,h}(y)$. For $M > 0$, let [compare with (3.6)–(3.7)]

$$
\begin{aligned}
\Phi_{\beta,h}^M(Q) &= \int_{\widetilde{E}} (\tilde{\pi}_1 Q)(dy)[\chi(y) \wedge M], \\
\overline{\Phi}_{\beta,h}^M(Q) &= \int_{\widetilde{E}} (\tilde{\pi}_1 Q)(dy)\chi(y) 1_{\{\chi(y)>M\}}.
\end{aligned}
\tag{B.4}
$$

Since $\phi_{\beta,h} \geq \frac{1}{2}$, $Q \mapsto \Phi_{\beta,h}^M(Q)$ is bounded and continuous. Our goal will be to compare $\overline{S}^{\mathrm{que}}(\beta, h; g)$ with its truncated analogue [compare with (3.15)–(3.16)]

$$
\overline{S}_M^{\mathrm{que}}(\beta, h; g) = \log \mathcal{N}(g) + \limsup_{N\to\infty} \frac{1}{N} \log E_g^*(e^{N\Phi_{\beta,h}^M(R_N^\omega)}), \qquad M > 0 \tag{B.5}
$$

and afterward let $M \to \infty$.

2. Note that

$$
\Phi_{\beta,h}(Q) - \overline{\Phi}_{\beta,h}^M(Q) \leq \Phi_{\beta,h}^M(Q). \tag{B.6}
$$

Therefore, for any $g > 0$ and any $-\infty < q < 0 < p < 1$ with $p^{-1} + q^{-1} = 1$, the reverse of Hölder's inequality gives

$$
\begin{aligned}
&E_g^*(e^{N\Phi_{\beta,h}^M(R_N^\omega)}) \\
&\quad\geq E_g^*(e^{N\Phi_{\beta,h}(R_N^\omega)} e^{-N\overline{\Phi}_{\beta,h}^M(R_N^\omega)}) \\
&\quad\geq E_g^*(e^{pN\Phi_{\beta,h}(R_N^\omega)})^{1/p} E_g^*(e^{-qN\overline{\Phi}_{\beta,h}^M(R_N^\omega)})^{1/q} \\
&\quad= E_g^*(e^{pN\Phi_{\beta,h}(R_N^\omega)})^{1/p} E_0^*(e^{-qN[\overline{\Phi}_{\beta,h}^M(R_N^\omega) - [g/(-q)] m_{R_N^\omega}]})^{1/q} \mathcal{N}(g)^{-N/q},
\end{aligned}
\tag{B.7}
$$

where the equality uses (2.4), and

$$
\begin{aligned}
N\overline{\Phi}^M_{\beta,h}(R_N^\omega) &= N\int_{\widetilde{E}}(\tilde{\pi}_1 R_N^\omega)(dy)\chi(y)1_{\{\chi(y)>M\}}\\
&= \sum_{i=1}^N \chi(y_i)1_{\{\chi(y_i)>M\}}, \qquad \text{(B.8)}\\
Nm_{R_N^\omega} &= \sum_{i=1}^N \tau(y_i).
\end{aligned}
$$

We next claim that ω-a.s. there exists an $M'(\omega)<\infty$, depending on β, h, g and p, such that

$$
E_0^*\big(e^{-qN[\overline{\Phi}^M_{\beta,h}(R_N^\omega)-[g/(-q)]m_{R_N^\omega}]}\big)\le 1 \qquad \forall M>M'(\omega). \tag{B.9}
$$

Indeed, $\{\chi(y_i)>M\}=\{-2\beta\sum_{k\in I_i}(\omega_k+h)>\log(2e^M-1)\}$, and so we can repeat the argument in the proof of Lemma A.1, restricting the estimates to m-values with $m\ge\log(2e^M-1)$. Clearly, there exists an $M_0<\infty$ such that $-[g/(-q)]Cm^2+(m+1)\le 0$ for $m\ge M_0$. Therefore, the claim in (B.9) follows for any $M'(\omega)$ such that $\log(2e^{M'(\omega)}-1)>M_0\vee M(\omega)$ with $M(\omega)$ defined below (A.5). With this choice of $M'(\omega)$, the term with $m<M(\omega)$ is absent, and we can estimate $\overline{\Phi}^M_{\beta,h}(R_N^\omega)-[g/(-q)]m_{R_N^\omega}\le 0$ as in (A.6)–(A.9).

3. We next apply Varadhan's lemma to (B.5) using Theorem 2.3 and the fact that $\Phi^M_{\beta,h}$ is bounded and continuous on $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$. This gives

$$
\begin{aligned}
\overline{S}^{\mathrm{que}}_M(\beta,h;g) &= \log\mathcal{N}(g)+\sup_{Q\in\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})}\big[\Phi^M_{\beta,h}(Q)-I^{\mathrm{que}}_g(Q)\big]\\
&= \sup_{Q\in\mathcal{R}}\big[\Phi^M_{\beta,h}(Q)-gm_Q-I^{\mathrm{ann}}(Q)\big]\\
&= \sup_{Q\in\mathcal{C}^{\mathrm{fin}}\cap\mathcal{R}}\big[\Phi^M_{\beta,h}(Q)-gm_Q-I^{\mathrm{ann}}(Q)\big] \qquad \text{(B.10)}\\
&\le \sup_{Q\in\mathcal{C}^{\mathrm{fin}}\cap\mathcal{R}}\big[\Phi_{\beta,h}(Q)-gm_Q-I^{\mathrm{ann}}(Q)\big]\\
&= S^{\mathrm{que}}(\beta,h;g),
\end{aligned}
$$

where the second equality uses (2.10) and (2.15), and the third equality uses that $\Phi^M_{\beta,h}\le M<\infty$ in combination with the fact that the Q's with $I^{\mathrm{ann}}(Q)=\infty$ or $m_Q=\infty$ do not contribute to the supremum. The inequality uses that $\Phi^M_{\beta,h}\le\Phi_{\beta,h}$. Recalling (3.15)–(3.16), combining (B.5) and (B.7)–(B.10), and letting $N\to\infty$ followed by $M\to\infty$, we get ($-q^{-1}+1=p^{-1}$)

$$
\begin{aligned}
&\Big(\frac{1}{p}-1\Big)\log\mathcal{N}(g)+\limsup_{N\to\infty}\frac{1}{pN}\log E_g^*\big(e^{pN\Phi_{\beta,h}(R_N^\omega)}\big)\\
&\qquad\le S^{\mathrm{que}}(\beta,h;g).
\end{aligned} \tag{B.11}
$$

4. It remains to show that the left-hand side of (B.11) tends to $\overline{S}^{\text{que}}(\beta, h; g)$ as $p \uparrow 1$. Define

$$S^{\beta,h}(p) = \limsup_{N\to\infty} \frac{1}{N} \log E_g^*\bigl(e^{pN\Phi_{\beta,h}(R_N^{\omega})}\bigr), \qquad p \geq 0. \tag{B.12}$$

Clearly, $p \mapsto S^{\beta,h}(p)$ is convex on $(0,\infty)$. To see that it is finite on $(0,\infty)$, note that

$$S^{\beta,h}(p) \leq \begin{cases} pS^{\beta,h}(1), & \text{for } p \in (0,1], \\ S^{p\beta,h}(1), & \text{for } p \in [1,\infty). \end{cases} \tag{B.13}$$

The first line follows from Jensen's inequality, and the second line from the fact that $p\Phi_{\beta,h} \leq \Phi_{p\beta,h}$ for $p \in [1,\infty)$ [recall (3.7)]. But we know from the remark made at the end of Section 6.1 that $S^{\beta,h}(1) = \overline{S}^{\text{que}}(\beta, h; g) - \log \mathcal{N}(g) < \infty$ because $g > 0$. Hence, $p \mapsto S^{\beta,h}(p)$ indeed is finite on $(0,\infty)$. Therefore, convexity implies continuity, and so $\lim_{p\uparrow 1} S^{\beta,h}(p) = S^{\beta,h}(1)$. Since the left-hand side of (B.11) equals $(p^{-1} - 1)\log \mathcal{N}(g) + p^{-1}S^{\beta,h}(p)$, the claim follows. $\square$

APPENDIX C: CONTINUITY AT $g = 0$

In this appendix, we prove (3.23). The key is the following proposition relating the two quenched LDP's in Theorem 2.3. Recall (2.12)–(2.14), and abbreviate $\mathcal{R}^{\text{fin}} = \{Q \in \mathcal{R} : m_Q < \infty\}$.

PROPOSITION C.1. *Suppose that E is finite. Then for every $Q \in \mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}})$ there exists a sequence (Q_n) in $\mathcal{R}^{\text{fin}}$ such that Q_n converges weakly to Q and* $\lim_{n\to\infty} I^{\text{ann}}(Q_n) = I^{\text{que}}(Q)$.

PROOF. The proof is *not self-contained*, because it uses the approximation argument in Birkner, Greven and den Hollander [4], Sections 3–4 (this argument was also exploited in Cheliotis and den Hollander [14], Appendix B). For simplicity, we pretend that the support of ρ is $\mathbb{N}$. The proof is easily extended to ρ with infinite support.

1. Recall the notation introduced in Section 2.1. We first assume that $Q \in \mathcal{P}^{\text{erg,fin}}(\widetilde{E}^{\mathbb{N}})$ with

$$\mathcal{P}^{\text{erg,fin}}(\widetilde{E}^{\mathbb{N}}) = \bigl\{Q \in \mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}}) : Q \text{ is ergodic}, m_Q < \infty\bigr\}. \tag{C.1}$$

For $M \in \mathbb{N}$ and $\epsilon_1 > 0$, choose

$$\mathcal{A} = \{z_a : a = 1, \dots, A\} \subset \widetilde{E}^M, \qquad \mathcal{B} = \bigl\{\zeta^{(b)} : b = 1, \dots, B\bigr\} = \kappa(\mathcal{A}) \tag{C.2}$$

as in [4], equations (3.5)–(3.6), satisfying also [4], equations (4.2)–(4.3) for a small neighborhood of Q. Each element of $\mathcal{B} \subset \widetilde{E}$ consists of approximately $L = Mm_Q$

letters (for simplicity we pretend that each $b \in \mathcal{B}$ has precisely L letters). Cut X into L-blocks, and let

$$G_j = 1_{\{\text{an element of } \mathcal{B} \text{ appears in } X|_{((j-1)L, jL]}\}}. \tag{C.3}$$

Note that (G_j) are i.i.d. Bernoulli(p) random variables with

$$p = p(M, \epsilon_1) = \exp[-M m_Q H(\Psi_Q | \nu^{\otimes \mathbb{N}})[1 + o(1)]], \qquad M \to \infty, \epsilon_1 \downarrow 0. \tag{C.4}$$

Therefore,

$$\sigma_1 = \min\{j \in \mathbb{N} : G_j = 1\} \tag{C.5}$$

is geometrically distributed with success probability p. Put $\widetilde{Y}_1 = \kappa(X|_{(0,(\sigma_1-1)L]})$ (and make a trivial modification when $\sigma_1 = 1$ to avoid an empty word later on). Given $X|_{((\sigma_1-1)L, \sigma_1 L]} = \zeta^{(b)} \in \mathcal{B}$, let

$$(\widetilde{Y}_2, \ldots, \widetilde{Y}_{M+1}) = z_a \tag{C.6}$$

be a suitably drawn random element of $\mathcal{A}$ [a is drawn uniformly from $\{a' : \kappa(z_{a'}) = \zeta^{(b)}\}$]. Repeating this construction, we obtain a random sequence $\widetilde{Y} = (\widetilde{Y}_j)$ in $\widetilde{E}^{\mathbb{N}}$. Denote the law of this random sequence by $\widetilde{Q}_{M,\epsilon_1}$. Note that, by construction, $\kappa(\widetilde{Y}) = X$, so that $\widetilde{Q}_{M,\epsilon_1} \in \mathcal{R}$, and that the consecutive $(M+1)$-blocks

$$(\widetilde{Y}_{(k-1)(M+1)+1}, \ldots, \widetilde{Y}_{k(M+1)})_{k \in \mathbb{N}} \tag{C.7}$$

form an i.i.d. sequence (in particular, $\widetilde{Q}_{M,\epsilon_1}$ is mixing and has finite mean word lengths). Furthermore, $\widetilde{Y}_1$ and $(\widetilde{Y}_2, \ldots, \widetilde{Y}_{M+1})$ are independent. Let $\widehat{Q}_{M,\epsilon_1}$ be the shift-invariant version of $\widetilde{Q}_{M,\epsilon_1}$ obtained by randomizing the position of the origin at the word level (not the letter level, as for Ψ_Q). Then

$$\widehat{Q}_{M,\epsilon_1} \in \mathcal{R}^{\mathrm{fin}}. \tag{C.8}$$

By construction, $\widehat{Q}_{M,\epsilon_1} \to Q$ weakly as $M \to \infty$ and $\epsilon_1 \downarrow 0$.

2. It remains to check that

$$I^{\mathrm{ann}}(\widehat{Q}_{M,\epsilon_1}) \to I^{\mathrm{que}}(Q), \qquad M \to \infty, \epsilon_1 \downarrow 0. \tag{C.9}$$

Since $\widehat{Q}_{M,\epsilon_1}$ is the shift-invariant mean of $\widetilde{Q}_{M,\epsilon_1}$, we have

$$\begin{aligned} &H(\widehat{Q}_{M,\epsilon_1} | q_{\rho,\nu}^{\otimes \mathbb{N}}) \\ &\quad = H(\widetilde{Q}_{M,\epsilon_1} | q_{\rho,\nu}^{\otimes \mathbb{N}}) \\ &\quad = \frac{1}{M+1} [h(\mathcal{L}(\widetilde{Y}_1) | q_{\rho,\nu}) + h(\mathcal{L}(\widetilde{Y}_2, \ldots, \widetilde{Y}_{M+1}) | q_{\rho,\nu}^{\otimes M})], \end{aligned} \tag{C.10}$$

where the second equality uses the special block structure of $\widetilde{Q}_{M,\epsilon_1}$. By construction, we have

$$h(\mathcal{L}(\widetilde{Y}_2, \ldots, \widetilde{Y}_{M+1}) | q_{\rho,\nu}^{\otimes M}) \in M H(Q | q_{\rho,\nu}^{\otimes \mathbb{N}}) + (-4\epsilon_1 M, 4\epsilon_1 M) \tag{C.11}$$

(see [4], equations (3.6) and (3.8)). Furthermore,

$$h(\mathcal{L}(\widetilde{Y}_1)|q_{\rho,\nu}) \in M(\alpha-1)m_Q H(\Psi_Q|\nu^{\otimes\mathbb{N}}) + (-\delta M, \delta M), \tag{C.12}$$

where $\delta \downarrow 0$ as $\epsilon_1 \downarrow 0$. To see why the latter holds, note that (below we write tL instead of $2 \vee tL$ to shorten the notation)

$$\begin{aligned}
&h(\mathcal{L}(\widetilde{Y}_1)|q_{\rho,\nu})\\
&\quad= \sum_{t=0}^{\infty} \sum_{\substack{x_1,\dots,x_{tL}\in E\\ \text{no } L\text{-block from } \mathcal{B}}} \frac{p(1-p)^t \prod_{k=1}^{tL} \nu(x_k)}{(1-p)^t}\\
&\qquad\qquad \times \log\left[\frac{p(1-p)^t((\prod_{k=1}^{tL}\nu(x_k))/(1-p)^t)}{\rho(tL)\prod_{k=1}^{tL}\nu(x_k)}\right]\\
&\quad= \sum_{t=0}^{\infty} p \sum_{\substack{x_1,\dots,x_{tL}\in E\\ \text{no } L\text{-block from } \mathcal{B}}} \left(\prod_{k=1}^{tL}\nu(x_k)\right)\log\left[\frac{p}{\rho(tL)}\right] \qquad \text{(C.13)}\\
&\quad= \sum_{t=0}^{\infty} p(1-p)^t \log\left[\frac{p}{\rho(tL)}\right]\\
&\quad= \log p - \sum_{t=0}^{\infty} p(1-p)^t \log(tL)\frac{\log(\rho(tL))}{\log(tL)}\\
&\quad= \log p + \alpha[1+o(1)]\sum_{t=0}^{\infty} p(1-p)^t \log(tL), \qquad L\to\infty.
\end{aligned}$$

Finally, note that $\log L = \log(Mm_Q) = O(\log M) = o(M)$ as $M \to \infty$ and

$$\begin{aligned}
\sum_{t=1}^{\infty} p(1-p)^t \log t &= -\log p + \sum_{t=1}^{\infty} p(1-p)^t \log(tp)\\
&= -\log p + \int_0^{\infty} e^{-y}\log y\,dy + o(1), \qquad p\downarrow 0,
\end{aligned} \tag{C.14}$$

where the integral equals minus Euler's constant. Since

$$-\log p \in Mm_Q H(\Psi_Q|\nu^{\otimes\mathbb{N}}) + [-\delta M, \delta M],$$

equations (C.13)–(C.14) combine to yield (C.12). Clearly, (C.10)–(C.12) imply (C.9), which completes the proof for $Q \in \mathcal{P}^{\mathrm{erg,fin}}(\widetilde{E}^{\mathbb{N}})$.

3. If $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ is ergodic with $m_Q = \infty$, then we approximate Q by $[Q]_{\mathrm{tr}}$ [recall (2.12)], approximate each $[Q]_{\mathrm{tr}}$ from inside $\mathcal{R}^{\mathrm{fin}}$ as above, and then diagonalize the approximation scheme. This yields the claim because $[Q]_{\mathrm{tr}} \to Q$ weakly and $I^{\mathrm{que}}([Q]_{\mathrm{tr}}) \to I^{\mathrm{que}}(Q)$ as $\mathrm{tr} \to \infty$ [recall (2.19)]. Finally, if $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ is

not ergodic, then we first approximate its ergodic decomposition by a finite sum and afterward approximate each summand as above (similarly as in [3], proof of Proposition 2, and [4], proof of Proposition 4.1). □

We are now ready to prove (3.23). Recall that $E=\mathbb{R}$. For $M\in\mathbb{N}$, let

$$D_M=\{-M,-M+1/M,\dots,M-1/M,M\} \tag{C.15}$$

be the grid of spacing $1/M$ in $[-M,M]$, which serves as a finite set of letters approximating E. Let $\widetilde{D}_M=\bigcup_{n\in\mathbb{N}}D_M^n$ be the set of finite words drawn from D_M. Let $T_M:E\to D_M$ be the letter map

$$T_M(x)=\begin{cases} M, & \text{for } x\in[M,\infty),\\ \lceil xM\rceil/M, & \text{for } x\in(-M,M),\\ -M, & \text{for } x\in(-\infty,-M]\end{cases} \tag{C.16}$$

and $\widetilde{T}_M:\widetilde{E}=\bigcup_{k\in\mathbb{N}}E^k\to[\widetilde{D}_M]_M=\bigcup_{k=1}^M D_M^k$ the word map

$$\begin{aligned}\widetilde{T}_M(y)=\widetilde{T}_M(x_1,\dots,x_m)&=(T_Mx_1,\dots,T_Mx_{m\wedge M}),\\ &\qquad m\in\mathbb{N},x_1,\dots,x_m\in E.\end{aligned} \tag{C.17}$$

For $Q\in\mathcal{P}^{\mathrm{inv}}(\widetilde{D}_M^{\mathbb{N}})$, define [compare with (3.6)–(3.7)]

$$\Phi_{\beta,h}^M(Q)=\int_{\widetilde{D}_M}(\tilde{\pi}_1Q)(dy)\log\phi_{\beta,h}^M(y), \tag{C.18}$$

where, for $y\in\widetilde{D}_M^{\mathbb{N}}$,

$$\phi_{\beta,h}^M(y)=\begin{cases}\phi_{\beta,h}(y), & \text{for } y=(x_1,\dots,x_m)\in\bigl[D_M\setminus\{-M,M\}\bigr]^m,\\ & m=1,\dots,M-1,\\ \frac{1}{2}, & \text{otherwise.}\end{cases} \tag{C.19}$$

Next, let $I_M^{\mathrm{que}}:\mathcal{P}^{\mathrm{inv}}(\widetilde{D}_M^{\mathbb{N}})\to[0,\infty]$ and $I_M^{\mathrm{ann}}:\mathcal{P}^{\mathrm{inv}}(\widetilde{D}_M^{\mathbb{N}})\to[0,\infty]$ be the quenched, respectively, annealed rate function when the disorder distribution is ν_M given by $\nu_M=\nu\circ T_M^{-1}$ and the word length distribution is ρ_M given by $\rho_M(m)=\rho(m)$ for $m=1,\dots,M-1$ and $\rho(M)=\sum_{m\geq M}\rho(m)$. Define

$$\mathcal{C}_M^{\mathrm{fin}}=\bigl\{Q\in\mathcal{P}^{\mathrm{inv}}\bigl(\widetilde{D}_M^{\mathbb{N}}\bigr):I_M^{\mathrm{ann}}(Q)<\infty,m_Q<\infty\bigr\}. \tag{C.20}$$

We know from (3.10), (3.13) and (3.16) that $\overline{S}^{\mathrm{que}}(\beta,h;g)$ is nonincreasing as a function of the disorder distribution ν. Inside the interval $(-M,M)$ the map T_M moves points upward, while $\phi_{\beta,h}(y)\geq\frac{1}{2}$, $y\in\widetilde{E}$. We therefore see from (C.19) that $\phi_{\beta,h}(y)\geq\phi_{\beta,h}^M(\widetilde{T}_My)$, $y\in\widetilde{E}$. Hence, $\overline{S}^{\mathrm{que}}(\beta,h;g)$ is bounded from below by its analogue $\overline{S}_M^{\mathrm{que}}(\beta,h;g)$ with ν replaced by ν_M, ρ by ρ_M and $\phi_{\beta,h}$ by $\phi_{\beta,h}^M$.

It therefore follows from the equality $\overline{S}^{\mathrm{que}}(\beta,h;g)=S^{\mathrm{que}}(\beta,h;g)$, $g\in(0,\infty)$, shown in Appendix B [recall (3.20)] and the first inequality that

$$
\begin{aligned}
\overline{S}^{\mathrm{que}}(\beta,h;0+) &\ge \overline{S}^{\mathrm{que}}_M(\beta,h;0+)\ge S^{\mathrm{que}}_M(\beta,h;0)\\
&= \sup_{Q\in\mathcal{C}^{\mathrm{fin}}_M\cap\mathcal{R}}\left[\Phi^M_{\beta,h}(Q)-I^{\mathrm{ann}}_M(Q)\right]\\
&= \sup_{Q\in\mathcal{C}^{\mathrm{fin}}_M}\left[\Phi^M_{\beta,h}(Q)-I^{\mathrm{que}}_M(Q)\right],
\end{aligned}
\tag{C.21}
$$

where the last equality in (C.21) uses Proposition C.1 in combination with the fact that D_M is finite and $\Phi^M_{\beta,h}$ is bounded and continuous on $\mathcal{C}^{\mathrm{fin}}_M$ [note that $I^{\mathrm{ann}}_M=I^{\mathrm{que}}_M$ on $\mathcal{C}^{\mathrm{fin}}_M\cap\mathcal{R}$ by (2.18)]. For $Q\in\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$, let $[Q]_M=Q\circ(\widetilde{T}^{\mathbb{N}}_M)^{-1}$. Then the right-hand side of (C.21) equals

$$
\sup_{Q\in\mathcal{C}^{\mathrm{fin}}}\left[\Phi^M_{\beta,h}\big([Q]_M\big)-I^{\mathrm{que}}_M\big([Q]_M\big)\right].
\tag{C.22}
$$

Next, $m_{[Q]_M}\le m_Q$, $\widetilde{T}_M$ is a projection, and relative entropies are nonincreasing under the action of a projection. Recalling (2.16)–(2.17), we therefore have $I^{\mathrm{que}}_M([Q]_M)\le I^{\mathrm{que}}(Q)$ for all $Q\in\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ and $M\in\mathbb{N}$. Hence, (C.21)–(C.22) combine to give

$$
\overline{S}^{\mathrm{que}}(\beta,h;0+)\ge\sup_{Q\in\mathcal{C}^{\mathrm{fin}}}\left[\Phi^M_{\beta,h}\big([Q]_M\big)-I^{\mathrm{que}}(Q)\right].
\tag{C.23}
$$

Finally, because $\lim_{M\to\infty}[Q]_M=Q$ weakly for all $Q\in\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ and $\lim_{M\to\infty}\phi^M_{\beta,h}(y)=\phi_{\beta,h}(y)$ for all $y\in\widetilde{E}$, Fatou's lemma tells us that $\lim_{M\to\infty}\Phi^M_{\beta,h}([Q]_M)\ge\Phi_{\beta,h}(Q)$. Hence, we arrive at [recall (3.8)]

$$
\overline{S}^{\mathrm{que}}(\beta,h;0+)\ge\sup_{Q\in\mathcal{C}^{\mathrm{fin}}}\left[\Phi_{\beta,h}(Q)-I^{\mathrm{que}}(Q)\right]=S^{\mathrm{que}}_*(\beta,h).
\tag{C.24}
$$

APPENDIX D: CONCENTRATION OF MEASURE ESTIMATES FOR THE DISORDER

First, we introduce some notation. After that, we state and prove the concentration of measure estimate for the disorder ω that was used in the proof of Lemmas A.1 and A.3 (Lemmas D.1–D.2 below).

Recall (1.2). The cumulant generating function $\lambda\mapsto M(\lambda)$ is analytic, nonnegative and strictly convex on $\mathbb{R}$, with $M(0)=M'(0)=0$ [recall (1.1)]. In particular, $G=M'$ and its inverse $H=G^{-1}$ are both analytic and strictly increasing on $[0,\infty)$. For a qualitative picture of M see Figure 10 below.

For $W,x>0$, define

$$
f_{W,x}(\lambda)=x\left[M(\lambda)-\lambda\frac{W}{x}\right],\qquad\lambda\in\mathbb{R}
\tag{D.1}
$$

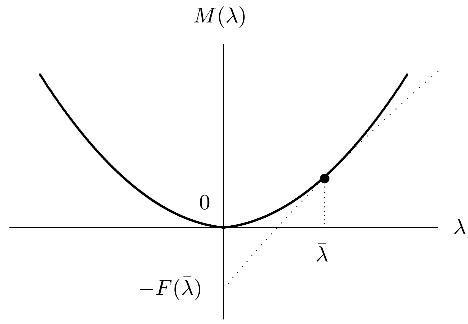

FIG. 10. *Qualitative picture of* $\lambda \mapsto M(\lambda)$. *The slope at* $\bar{\lambda}$ *equals* $G(\bar{\lambda})$.

and note that $\lambda \mapsto f_{W,x}(\lambda)$ is strictly convex on $\mathbb{R}$, with $f_{W,x}(0) = 0$ and $f'_{W,x}(0) = -W < 0$. Putting

$$\chi = \lim_{\lambda \to \infty} G(\lambda) \in (0, \infty] \tag{D.2}$$

[which, by (1.2), equals the supremum of the support of the law of $-\omega_1$], we have $\lim_{\lambda \to \infty} f_{W,x}(\lambda)/\lambda = \chi x - W$, and so there are two cases:

(I) If $\frac{W}{x} \le \chi$, then $f_{W,x}$ has a unique minimizer at some $\lambda_* \in (0, \infty]$. Note that $\lambda^* = \infty$ if and only if $\frac{W}{x} = \chi$.

(II) If $\frac{W}{x} > \chi$, then $f_{W,x}$ attains it minimum at infinity. In this case $\lim_{\lambda \to \infty} f_{W,x}(\lambda) = (\chi x - W) \times \infty = -\infty$.

In case (I), we have

$$\lambda_* = \lambda_*(W, x) = H\left(\frac{W}{x}\right), \qquad f_{W,x}(\lambda_*) = -x\big[\lambda_* G(\lambda_*) - M(\lambda_*)\big]. \tag{D.3}$$

Since $H(y)$ is well defined only for $y \le \chi$, in what follows we will always assume that the arguments of H are at most χ.

Our concentration of measure estimate is the following. Let

$$F(\lambda) = \lambda G(\lambda) - M(\lambda), \qquad \lambda \in [0, \infty). \tag{D.4}$$

LEMMA D.1. *For* $n \in \mathbb{N}$ *and* $A, B > 0$,

$$\mathbb{P}\left(\sum_{k=1}^{n} \omega_k \le -A - nB\right) \begin{cases} \le \exp\left[-nF\left(H\left(\dfrac{A}{n} + B\right)\right)\right], \\ \qquad\qquad \textit{when } A/n + B \le \chi, \\ = 0, \qquad \textit{when } A/n + B > \chi, \end{cases} \tag{D.5}$$

where

$$nF\left(H\left(\frac{A}{n} + B\right)\right) \ge C(A + n) \qquad \textit{when } A/n + B \le \chi \tag{D.6}$$

with

$$C = \tfrac{1}{2}\big[F\big(H(B)\big) \wedge F\big(H(1)\big)\big] > 0. \tag{D.7}$$

PROOF. Estimate

$$\begin{aligned}
\mathbb{P}\left(\sum_{k=1}^{n} \omega_k \le -W\right) &= \inf_{\lambda>0} \mathbb{P}\big(e^{-\lambda \sum_{k=1}^{n} \omega_k} \ge e^{\lambda W}\big) \\
&\le \inf_{\lambda>0} e^{-\lambda W}\big[\mathbb{E}\big(e^{-\lambda\omega_1}\big)\big]^n \\
&= \inf_{\lambda>0} e^{-\lambda W + nM(\lambda)} = e^{\inf_{\lambda>0} f_{W,n}(\lambda)}
\end{aligned} \tag{D.8}$$

with $\lambda \mapsto f_{W,n}(\lambda)$ the function defined in (D.1). In case (I), (D.3) shows that the minimal value of $f_{W,n}$ is $-nF(\lambda_*(W,n)) = -nF(H(\frac{W}{n}))$. Together with the lower bound on $nF(H(\frac{W}{n}))$ that is derived in Lemma D.2 below, this proves the first line of (D.5) with the estimates in (D.6)–(D.7). In case (II), $f_{W,n}$ attains its infimum at infinity, with $f_{W,n}(\infty) = -\infty$, which proves the second line of (D.5). □

LEMMA D.2. *For every $A, B > 0$ and $x \in [1,\infty)$ with $A/x + B \le \chi$ there exists a $C > 0$ (depending on B only) such that*

$$xF\left(H\left(\frac{A}{x} + B\right)\right) \ge C(A + x), \qquad x \in [1, \infty). \tag{D.9}$$

PROOF. For $x \ge A$, estimate

$$xF\left(H\left(\frac{A}{x} + B\right)\right) \ge xF\big(H(B)\big) \ge \frac{1}{2}(A + x)F\big(H(B)\big). \tag{D.10}$$

For $x \le A$, on the other hand, estimate

$$\begin{aligned}
xF\left(H\left(\frac{A}{x} + B\right)\right) &\ge A\left(\frac{A}{x}\right)^{-1} F\left(H\left(\frac{A}{x}\right)\right) \\
&\ge AF\big(H(1)\big) \ge \frac{1}{2}(A + x)F\big(H(1)\big),
\end{aligned} \tag{D.11}$$

where the second inequality uses that $y \mapsto y^{-1}F(H(y))$ is strictly increasing on $(0, \chi)$. Combining the two estimates, we get the claim with C given by (D.7). Note that $C > 0$ because $H(0) = 0$ and $F(H(0)) = 0$. □

Acknowledgments. F. den Hollander thanks M. Birkner and F. Redig for fruitful discussions. The authors thank two anonymous referees for their careful reading of the manuscript and for detailed comments.

REFERENCES

[1] BERGER, Q., CARAVENNA, F., POISAT, J., SUN, R. and ZYGOURAS, N. (2014). The critical curves of the random pinning and copolymer models at weak coupling. *Comm. Math. Phys.* **326** 507–530. MR3165465

[2] BINGHAM, N. H., GOLDIE, C. M. and TEUGELS, J. L. (1987). *Regular Variation. Encyclopedia of Mathematics and Its Applications* **27**. Cambridge Univ. Press, Cambridge. MR0898871

[3] BIRKNER, M. (2008). Conditional large deviations for a sequence of words. *Stochastic Process. Appl.* **118** 703–729. MR2411517

[4] BIRKNER, M., GREVEN, A. and DEN HOLLANDER, F. (2010). Quenched large deviation principle for words in a letter sequence. *Probab. Theory Related Fields* **148** 403–456. MR2678894

[5] BIRKNER, M., GREVEN, A. and DEN HOLLANDER, F. (2011). Collision local time of transient random walks and intermediate phases in interacting stochastic systems. *Electron. J. Probab.* **16** 552–586. MR2786642

[6] BISKUP, M. and DEN HOLLANDER, F. (1999). A heteropolymer near a linear interface. *Ann. Appl. Probab.* **9** 668–687. MR1722277

[7] BODINEAU, T. and GIACOMIN, G. (2004). On the localization transition of random copolymers near selective interfaces. *J. Stat. Phys.* **117** 801–818. MR2107896

[8] BODINEAU, T., GIACOMIN, G., LACOIN, H. and TONINELLI, F. L. (2008). Copolymers at selective interfaces: New bounds on the phase diagram. *J. Stat. Phys.* **132** 603–626. MR2429695

[9] BOLTHAUSEN, E. and DEN HOLLANDER, F. (1997). Localization transition for a polymer near an interface. *Ann. Probab.* **25** 1334–1366. MR1457622

[10] CARAVENNA, F. and GIACOMIN, G. (2005). On constrained annealed bounds for pinning and wetting models. *Electron. Commun. Probab.* **10** 179–189 (electronic). MR2162817

[11] CARAVENNA, F. and GIACOMIN, G. (2010). The weak coupling limit of disordered copolymer models. *Ann. Probab.* **38** 2322–2378. MR2683632

[12] CARAVENNA, F., GIACOMIN, G. and GUBINELLI, M. (2006). A numerical approach to copolymers at selective interfaces. *J. Stat. Phys.* **122** 799–832. MR2213950

[13] CARAVENNA, F., GIACOMIN, G. and TONINELLI, F. L. (2012). Copolymers at selective interfaces: Settled issues and open problems. In *Probability in Complex Physical Systems. Proceedings in Mathematics* **11** 289–310. Springer, Berlin.

[14] CHELIOTIS, D. and DEN HOLLANDER, F. (2013). Variational characterization of the critical curve for pinning of random polymers. *Ann. Probab.* **41** 1767–1805. MR3098058

[15] DEMBO, A. and ZEITOUNI, O. (1998). *Large Deviations Techniques and Applications*, 2nd ed. *Applications of Mathematics* (*New York*) **38**. Springer, New York. MR1619036

[16] DEN HOLLANDER, F. (2009). *Random Polymers. Lecture Notes in Math.* **1974**. Springer, Berlin. MR2504175

[17] DEN HOLLANDER, F. (2010). A key large deviation principle for interacting stochastic systems. In *Proceedings of the International Congress of Mathematicians. Volume IV* 2258–2274. Hindustan Book Agency, New Delhi. MR2827970

[18] DEN HOLLANDER, F. and OPOKU, A. A. (2013). Copolymer with pinning: Variational characterization of the phase diagram. *J. Stat. Phys.* **152** 846–893. MR3101486

[19] FELLER, V. (1968). *An Introduction to Probability Theory and Its Applications*, 3rd ed. Wiley, New York.

[20] GAREL, T., HUSE, D. A., LEIBLER, S. and ORLAND, H. (1989). Localization transition of random chains at interfaces. *Europhys. Lett.* **8** 9–13.

[21] GIACOMIN, G. (2007). *Random Polymer Models*. Imperial College Press, London. MR2380992

[22] GIACOMIN, G. and TONINELLI, F. L. (2005). Estimates on path delocalization for copolymers at selective interfaces. *Probab. Theory Related Fields* **133** 464–482. MR2197110

[23] GIACOMIN, G. and TONINELLI, F. L. (2006). Smoothing of depinning transitions for directed polymers with quenched disorder. *Phys. Rev. Lett.* **96** 070602.

[24] GIACOMIN, G. and TONINELLI, F. L. (2006). Smoothing effect of quenched disorder on polymer depinning transitions. *Comm. Math. Phys.* **266** 1–16. MR2231963

[25] GIACOMIN, G. and TONINELLI, F. L. (2006). The localized phase of disordered copolymers with adsorption. *ALEA Lat. Am. J. Probab. Math. Stat.* **1** 149–180. MR2249653

[26] MOURRAT, J.-C. (2012). On the delocalized phase of the random pinning model. In *Séminaire de Probabilités XLIV* 401–407. Springer, Heidelberg. MR2953357

[27] ORLANDINI, E., RECHNITZER, A. and WHITTINGTON, S. G. (2002). Random copolymers and the Morita approximation: Polymer adsorption and polymer localization. *J. Phys. A* **35** 7729–7751. MR1947129

[28] TONINELLI, F. L. (2008). Disordered pinning models and copolymers: Beyond annealed bounds. *Ann. Appl. Probab.* **18** 1569–1587. MR2434181

[29] TONINELLI, F. L. (2009). Coarse graining, fractional moments and the critical slope of random copolymers. *Electron. J. Probab.* **14** 531–547. MR2480552

E. BOLTHAUSEN
INSTITUT FÜR MATHEMATIK
UNIVERSITÄT ZÜRICH
WINTERTHURERSTRASSE 190
CH-8057 ZÜRICH
SWITZERLAND
E-MAIL: eb@math.uzh.ch

F. DEN HOLLANDER
MATHEMATICAL INSTITUTE
LEIDEN UNIVERSITY
P.O. BOX 9512
2300 RA LEIDEN
THE NETHERLANDS
E-MAIL: denholla@math.leidenuniv.nl

A. A. OPOKU
DELFT INSTITUTE OF APPLIED MATHEMATICS
ELECTRICAL ENGINEERING,
MATHEMATICS AND COMPUTER SCIENCE FACULTY
DELFT UNIVERSITY OF TECHNOLOGY
MEKELWEG 4
2628 CD DELFT
THE NETHERLANDS
E-MAIL: A.A.Opoku@tudelft.nl